Multiplication Table
(Quadratics Part I)

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Abstract: This study develops a geometric framework for integer multiplication based on the Full Multiplication Table (FMT) and its minimal representative, the Triangular Multiplication Table (TMT). The FMT reveals a global polynomial architecture: every diagonal sequence is of the form $y(y \pm L)$, exhibits constant second differences, and belongs to a unified family of square-minus-square and oblong-minus-oblong curves. These curves appear as parabolic bundles that share a common axis and encode the quadratic symmetries of multiplication. Restricting the FMT to the first octant produces the TMT, a domain that contains all essential multiplicative geometry while eliminating redundancy. Within the TMT, the square and oblong families intersect each odd column in a highly structured way. When the vertical range is restricted to $2 \leq y \leq \left\lfloor \frac{x}{4} \right\rfloor$, each column is either "covered'' by at least one square or oblong intersection (product) or "uncovered'' by both. These two geometric outcomes correspond exactly to composite and prime values of x, respectively. Thus, the classical arithmetic distinction between primes and composites emerges from a purely geometric covering mechanism driven by the quadratic structure of the multiplication table. The analysis establishes the parabolic loci of squares and oblongs, explains their shared symmetry axis $x = 4y$, and formulates the covering criterion that underlies the square–oblong sieve.

Keywords: Multiplication Table; Triangular Multiplication Table (TMT); Full Multiplication Table (FMT); Quadratic Sequences; Square-minus-square numbers; Oblong-minus-oblong numbers.

2020 Mathematics Subject Classification: 11B85 – Arithmetic progressions; sequences in general ; 11B83 – Special sequences and polynomials (e.g., square numbers, oblong numbers); 11A25 – Arithmetic functions; related numbers; 97F60 – Mathematics education: Number theory.

1. Introduction

This work develops a geometric framework for understanding integer multiplication and its quadratic structure. The Full Multiplication Table (FMT), represented as the infinite grid of products $x * y$ over the integer lattice, is commonly viewed as an elementary combinatorial object. Yet this grid hides a system of quadratic symmetries. Its diagonals follow second-degree polynomial laws; its hyperbolas trace curves of constant product; and its alternating bundles of square and oblong values form coherent parabolic families. Together, these features reveal multiplication as a highly structured quadratic geometry embedded in the integer plane.

The first aim of this study (part I) is to make this geometry explicit. The FMT naturally decomposes into diagonal sequences of the form $y(y + L)$, each with constant second differences equal to 2. These sequences include the classical perfect squares y^2 and the oblong numbers $y(y + 1)$ and extend to all even and odd offsets of the form $y(y \pm 2b)$ and $y\big(y \pm (2b + 1)\big)$. Their arrangement is both symmetric and exhaustive: every diagonal of the FMT is quadratic, and all such sequences arise from a single master formula that organizes the entire diagonal architecture. This provides a unified explanation for why squares, oblongs, and their systematic variants appear where they do in the multiplication table.

The second aim of this first part is to show that the Triangular Multiplication Table (TMT), obtained by restricting the FMT to the first octant $1 \leq y \leq x$, is sufficient to recover the full geometric structure of the multiplication table. Despite its apparent simplicity, the TMT contains every diagonal family, every hyperbola $xy = n$, and every quadratic difference sequence present in the FMT. Because each product appears only once in this reduced domain, the TMT provides an economical yet complete model of multiplicative geometry.

Most importantly, the TMT reveals a fundamental sieve structure. When the square and oblong families are restricted to the admissible vertical range $2 \leq y \leq \left\lfloor \frac{x}{4} \right\rfloor$, they interact with each odd column x in one of only two possible ways: either the column is “covered” by at least one square or oblong intersection (or product), or it is “uncovered” by both families. These two cases correspond exactly to composite and prime values of x, respectively. Thus, the classical arithmetic distinction between primes and composites emerges as a geometric covering criterion within the TMT lattice. This phenomenon generalizes the divisor-pair picture on a fixed hyperbola $xy = n$, replacing it by a dynamic family of square and oblong curves moving across the table.

The final objective of this introductory part (Quadratics Part I) is to establish the quadratic–parabolic foundation upon which later results will build. The parabolic loci of squares and oblongs are derived algebraically and geometrically, their shared axis of symmetry $x = 4y$ is explained, and their role in determining coverage versus non-coverage on each column is described in detail. This lays the groundwork for the higher-level analytic goals pursued in later chapters, particularly the geometric reformulation of classical prime conjectures such as Legendre’s and Oppermann’s.

The central theme is therefore twofold: (i) to expose the intrinsic quadratic geometry of multiplication, and (ii) to demonstrate how this geometry naturally organizes the positive integers into covered (composite) and uncovered (prime) columns within the TMT.

By presenting multiplication through its geometric symmetries (linear, hyperbolic, and parabolic) this study provides a unified framework that tie elementary arithmetic and analytic number theory.

1.1. Glossary of Colors, Mathematical Terms and Abbreviations

Please consult: Kusniec, Charles. (2023). Symbolic Conventions and Lexicon (Preprint). Zenodo. https://doi.org/10.5281/zenodo.10127987.

For real-time updates and immediate communication at https://drive.google.com/file/d/1Hh6mMclmEAbz3C7QVuEhJiy3uCqlF0lI/view?usp=sharing, or our Facebook group remains the primary platform for the latest modifications and discussions.

Each update to the document is promptly shared with the community at https://www.facebook.com/groups/snypo/posts/653023753021234/.

2. C000430 Full Multiplication Table (FMT): Structure and Interpretation

The full multiplication table (FMT) displays the products of integers across a symmetric grid centered at the origin. Each entry corresponds to the value $P(x, y) = xy$, where x is the horizontal coordinate and y the vertical coordinate. The row $y = 0$ and the column $x = 0$ consist entirely of zeros, forming the coordinate axes.

Each vertical column defines a linear function of y, given by $V_x(y) = xy$. Each horizontal row defines a linear function of x, given by $H_y(x) = yx$. Both rows and columns therefore follow arithmetic progressions, illustrating the bilinear structure of multiplication.

The diagonals highlight the role of squares: along the 45° line $x = y$ the entries are perfect squares x^2, while along the $-45°$ line $x = -y$ the entries are the negative squares $-x^2$. This basic geometric organization makes visible several core properties of multiplication, including symmetry, sign behavior, commutativity, and linearity.

a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
b	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
c	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	-100	-90	-80	-70	-60	-50	-40	-30	-20	-10	0	10	20	30	40	50	60	70	80	90	100
9	-90	-81	-72	-63	-54	-45	-36	-27	-18	-9	0	9	18	27	36	45	54	63	72	81	90
8	-80	-72	-64	-56	-48	-40	-32	-24	-16	-8	0	8	16	24	32	40	48	56	64	72	80
7	-70	-63	-56	-49	-42	-35	-28	-21	-14	-7	0	7	14	21	28	35	42	49	56	63	70
6	-60	-54	-48	-42	-36	-30	-24	-18	-12	-6	0	6	12	18	24	30	36	42	48	54	60
5	-50	-45	-40	-35	-30	-25	-20	-15	-10	-5	0	5	10	15	20	25	30	35	40	45	50
4	-40	-36	-32	-28	-24	-20	-16	-12	-8	-4	0	4	8	12	16	20	24	28	32	36	40
3	-30	-27	-24	-21	-18	-15	-12	-9	-6	-3	0	3	6	9	12	15	18	21	24	27	30
2	-20	-18	-16	-14	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12	14	16	18	20
1	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-1	10	9	8	7	6	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10
-2	20	18	16	14	12	10	8	6	4	2	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20
-3	30	27	24	21	18	15	12	9	6	3	0	-3	-6	-9	-12	-15	-18	-21	-24	-27	-30
-4	40	36	32	28	24	20	16	12	8	4	0	-4	-8	-12	-16	-20	-24	-28	-32	-36	-40
-5	50	45	40	35	30	25	20	15	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45	-50
-6	60	54	48	42	36	30	24	18	12	6	0	-6	-12	-18	-24	-30	-36	-42	-48	-54	-60
-7	70	63	56	49	42	35	28	21	14	7	0	-7	-14	-21	-28	-35	-42	-49	-56	-63	-70
-8	80	72	64	56	48	40	32	24	16	8	0	-8	-16	-24	-32	-40	-48	-56	-64	-72	-80
-9	90	81	72	63	54	45	36	27	18	9	0	-9	-18	-27	-36	-45	-54	-63	-72	-81	-90
-10	100	90	80	70	60	50	40	30	20	10	0	-10	-20	-30	-40	-50	-60	-70	-80	-90	-100

Figure C000430 Full Multiplication Table (FMT)

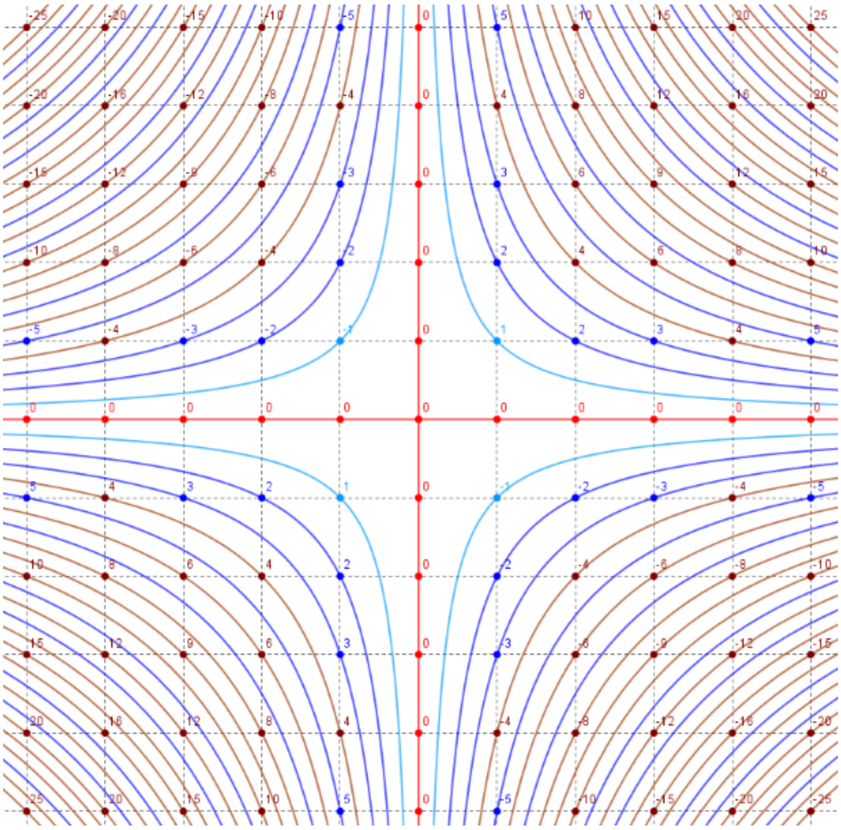


Figure C000430-1 Hyperbolas $xy = c$ in the full multiplication table (FMT)

The hyperbolas defined by $xy = c$ provide the continuous basic geometric background of the Full Multiplication Table. They express the bilinear structure of multiplication in smooth, symmetric curves across the plane. Arithmetic patterns such as perfect squares, oblongs, or prime-related gaps then appear as discrete highlights against this background: special alignments of lattice points that occur only at selected integer coordinates.

2.1. From Hyperbolas to Families of Squares and Oblongs

Among the many integer sequences that appear in the FMT, two quadratics play a central role: the perfect squares y^2 A000290 and the oblongs $(y-1)y$ A002378. These sequences arise as distinguished lattice points along the hyperbolas, forming the basic "square–oblong mechanism. Squares lie on the main diagonal, while oblongs appear at once next to them, both above and below (equivalently, on the left and right sides), corresponding to consecutive products of integers.

To generalize this structure, we introduce parametric families of quadratic forms:

- **Square family**: $y(y \pm 2b)$, which reduces to y^2 when $b = 0$.
- **Oblong family**: $y\big(y \pm (2b+1)\big)$, which reduces to $y(y \pm 1)$ when $b = 0$.

Each parameter $b \geq 0$ defines a paired system of parabolic bundles, one even-shifted (squares) and one odd-shifted (oblongs). Together they extend the classical square–oblong pair into a unified basic geometric framework for analyzing coverage and non-coverage within the FMT.

The FMT thus provides the foundational picture: multiplication as a symmetric bilinear structure expressed both as linear sequences along rows and columns and as hyperbolic curves across the plane. This perspective prepares the way for the TMT, obtained by restricting to the range $1 \leq y \leq x$, where each factor pairs are represented uniquely.

3. C003556 Full Multiplication Table Diagonals (FMTD): Diagonal Structure and Quadratic Patterns

Since each row and each column of the multiplication table follows a linear function, any diagonal progression that combines them necessarily arises from the interaction of two first-degree expressions. The product of such linear terms yields second-degree polynomials, ensuring that all $\pm 45°$ diagonals of the multiplication table correspond to quadratic sequences. This explains, from first principles, why squares, oblong numbers, and their systematic variants appear precisely along the diagonal directions.

L=		-4	L=		-3	L=		-2	L=		-1	L=		0	L=		1	L=		2	L=		3	L=		4
https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A002378	https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A005563	https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A028552	https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A028347	https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A028557	https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A028560	https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A028563	https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A028566	https://oeis.org/A256958	https://oeis.org/A256958	https://oeis.org/A028569
y	y+L	y((y+L)	y	y+L	y((y+L)	y	y+L	y((y+L)	y	y+L	y((y+L)	y	y+L	y((y+L)	y	y+L	y((y+L)	y	y+L	y((y+L)	y	y+L	y((y+L)	y	y+L	y((y+L)
10	6	60	10	7	70	10	8	80	10	9	90	10	10	100	10	11	110	10	12	120	10	13	130	10	14	140
9	5	45	9	6	54	9	7	63	9	8	72	9	9	81	9	10	90	9	11	99	9	12	108	9	13	117
8	4	32	8	5	40	8	6	48	8	7	56	8	8	64	8	9	72	8	10	80	8	11	88	8	12	96
7	3	21	7	4	28	7	5	35	7	6	42	7	7	49	7	8	56	7	9	63	7	10	70	7	11	77
6	2	12	6	3	18	6	4	24	6	5	30	6	6	36	6	7	42	6	8	48	6	9	54	6	10	60
5	1	5	5	2	10	5	3	15	5	4	20	5	5	25	5	6	30	5	7	35	5	8	40	5	9	45
4	0	0	4	1	4	4	2	8	4	3	12	4	4	16	4	5	20	4	6	24	4	7	28	4	8	32
3	-1	-3	3	0	0	3	1	3	3	2	6	3	3	9	3	4	12	3	5	15	3	6	18	3	7	21
2	-2	-4	2	-1	-2	2	0	0	2	1	2	2	2	4	2	3	6	2	4	8	2	5	10	2	6	12
1	-3	-3	1	-2	-2	1	-1	-1	1	0	0	1	1	1	1	2	2	1	3	3	1	4	4	1	5	5
0	-4	0	0	-3	0	0	-2	0	0	-1	0	0	0	0	0	1	0	0	2	0	0	3	0	0	4	0
-1	-5	5	-1	-4	4	-1	-3	3	-1	-2	2	-1	-1	1	-1	0	0	-1	1	-1	-1	2	-2	-1	3	-3
-2	-6	12	-2	-5	10	-2	-4	8	-2	-3	6	-2	-2	4	-2	-1	2	-2	0	0	-2	1	-2	-2	2	-4
-3	-7	21	-3	-6	18	-3	-5	15	-3	-4	12	-3	-3	9	-3	-2	6	-3	-1	3	-3	0	0	-3	1	-3
-4	-8	32	-4	-7	28	-4	-6	24	-4	-5	20	-4	-4	16	-4	-3	12	-4	-2	8	-4	-1	4	-4	0	0
-5	-9	45	-5	-8	40	-5	-7	35	-5	-6	30	-5	-5	25	-5	-4	20	-5	-3	15	-5	-2	10	-5	-1	5
-6	-10	60	-6	-9	54	-6	-8	48	-6	-7	42	-6	-6	36	-6	-5	30	-6	-4	24	-6	-3	18	-6	-2	12
-7	-11	77	-7	-10	70	-7	-9	63	-7	-8	56	-7	-7	49	-7	-6	42	-7	-5	35	-7	-4	28	-7	-3	21
-8	-12	96	-8	-11	88	-8	-10	80	-8	-9	72	-8	-8	64	-8	-7	56	-8	-6	48	-8	-5	40	-8	-4	32
-9	-13	117	-9	-12	108	-9	-11	99	-9	-10	90	-9	-9	81	-9	-8	72	-9	-7	63	-9	-6	54	-9	-5	45
-10	-14	140	-10	-13	130	-10	-12	120	-10	-11	110	-10	-10	100	-10	-9	90	-10	-8	80	-10	-7	70	-10	-6	60

Figure C003556 Full Multiplication Table Diagonals (FMTD)

The figure C003556 presents a systematic construction of diagonal sequences derived from the integer multiplication table (C000430 FMT). It illustrates how families of quadratic expressions naturally arise along the $\pm 45°$ diagonals.

Each vertical block of three columns in the figure is associated with a fixed integer offset L and consists of:

- the index y,
- the shifted index $y \pm L$,
- and the product $y(y \pm L)$.

This product generates the quadratic expression in y:

$$Y(y) = y(y \pm L) = y^2 \pm Ly$$

In this formulation, the parameter L decides the diagonal's offset from the central diagonal or axis of squares, while the $\pm$ sign produces the symmetric diagonals above and below the square diagonal. The two branches differ only by this offset, which varies proportionally with the coefficient of the linear term $\pm L$.

When $L = 0$, the formula reduces to y^2, the perfect squares. This is the only case without offset because the linear coefficient is 0.

When $L = 1$, the forms $y(y + 1)$ and $y(y - 1)$ give the classical oblong numbers, lying immediately above and below the squares.

For $L = 2$, the forms $y(y + 2)$ and $y(y - 2)$ generate quadratic sequences shifted two steps from the central diagonal.

In general, for each $L \geq 1$, the pair $y(y + L)$ and $y(y - L)$ defines symmetric families whose offset from the square diagonal grows in direct proportion to L.

Thus, the diagonal structure of the multiplication table is unified under the expression $y(y \pm L)$: all $\pm 45°$ diagonals are quadratic, all share the same leading coefficient 1, and all differ only by a systematic offset controlled by L. The squares are the special central case with zero linear term, while all other diagonals are symmetric shifts around them.

More generally, the leading coefficient of each quadratic sequence is determined by the slope of its diagonal progression. For the main $45°$ diagonal, the sequence is given by y^2 (coefficient 1). For diagonals inclined at slope ratios $1:2$ or $1:3$, the corresponding quadratic sequences have leading coefficients 2 and 3, i.e., asymptotically $2y^2$ and $3y^2$, respectively, with all parallel diagonals sharing the same quadratic coefficient. This reveals that the geometric inclination of a diagonal directly governs the growth rate of its quadratic sequence.

4. C000776 Square-minus-squares sequences: Diagonal Offsets of Perfect Squares

The figure C000776 presents a family of quadratic sequences of the form:

$$y(y \pm L) = y^2 \pm Ly$$

with L even. Explicitly,

$$y^2 \pm 2by = (y \pm b)^2 - b^2$$

Thus every even-offset diagonal corresponds to 'square minus a fixed square,' with the fixed square given by b^2. When $L = 0$ this reduces to the central diagonal of perfect squares y^2.

a	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	http://oeis.org/A000012
b	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	http://oeis.org/A000004
c	-100	-81	-64	-49	-36	-25	-16	-9	-4	-1	0	-1	-4	-9	-16	-25	-36	-49	-64	-81	-100	http://oeis.org/A000290
	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	
10	0	19	36	51	64	75	84	91	96	99	100	99	96	91	84	75	64	51	36	19	0	http://oeis.org/A120071
9	-19	0	17	32	45	56	65	72	77	80	81	80	77	72	65	56	45	32	17	0	-19	http://oeis.org/A098850
8	-36	-17	0	15	28	39	48	55	60	63	64	63	60	55	48	39	28	15	0	-17	-36	http://oeis.org/A098849
7	-51	-32	-15	0	13	24	33	40	45	48	49	48	45	40	33	24	13	0	-15	-32	-51	http://oeis.org/A098848
6	-64	-45	-28	-13	0	11	20	27	32	35	36	35	32	27	20	11	0	-13	-28	-45	-64	http://oeis.org/A098847
5	-75	-56	-39	-24	-11	0	9	16	21	24	25	24	21	16	9	0	-11	-24	-39	-56	-75	https://oeis.org/A098603
4	-84	-65	-48	-33	-20	-9	0	7	12	15	16	15	12	7	0	-9	-20	-33	-48	-65	-84	https://oeis.org/A028566
3	-91	-72	-55	-40	-27	-16	-7	0	5	8	9	8	5	0	-7	-16	-27	-40	-55	-72	-91	https://oeis.org/A028560
2	-96	-77	-60	-45	-32	-21	-12	-5	0	3	4	3	0	-5	-12	-21	-32	-45	-60	-77	-96	https://oeis.org/A028347
1	-99	-80	-63	-48	-35	-24	-15	-8	-3	0	1	0	-3	-8	-15	-24	-35	-48	-63	-80	-99	https://oeis.org/A005563
0	-100	-81	-64	-49	-36	-25	-16	-9	-4	-1	0	-1	-4	-9	-16	-25	-36	-49	-64	-81	-100	http://oeis.org/A000290
-1	-99	-80	-63	-48	-35	-24	-15	-8	-3	0	1	0	-3	-8	-15	-24	-35	-48	-63	-80	-99	https://oeis.org/A005563
-2	-96	-77	-60	-45	-32	-21	-12	-5	0	3	4	3	0	-5	-12	-21	-32	-45	-60	-77	-96	https://oeis.org/A028347
-3	-91	-72	-55	-40	-27	-16	-7	0	5	8	9	8	5	0	-7	-16	-27	-40	-55	-72	-91	https://oeis.org/A028560
-4	-84	-65	-48	-33	-20	-9	0	7	12	15	16	15	12	7	0	-9	-20	-33	-48	-65	-84	https://oeis.org/A028566
-5	-75	-56	-39	-24	-11	0	9	16	21	24	25	24	21	16	9	0	-11	-24	-39	-56	-75	https://oeis.org/A098603
-6	-64	-45	-28	-13	0	11	20	27	32	35	36	35	32	27	20	11	0	-13	-28	-45	-64	http://oeis.org/A098847
-7	-51	-32	-15	0	13	24	33	40	45	48	49	48	45	40	33	24	13	0	-15	-32	-51	http://oeis.org/A098848
-8	-36	-17	0	15	28	39	48	55	60	63	64	63	60	55	48	39	28	15	0	-17	-36	http://oeis.org/A098849
-9	-19	0	17	32	45	56	65	72	77	80	81	80	77	72	65	56	45	32	17	0	-19	http://oeis.org/A098850
-10	0	19	36	51	64	75	84	91	96	99	100	99	96	91	84	75	64	51	36	19	0	http://oeis.org/A120071
	http://oeis.org/A120071	http://oeis.org/A098850	http://oeis.org/A098849	http://oeis.org/A098848	http://oeis.org/A098847	https://oeis.org/A098603	https://oeis.org/A028566	https://oeis.org/A028560	https://oeis.org/A028347	https://oeis.org/A005563	http://oeis.org/A000290	https://oeis.org/A005563	https://oeis.org/A028347	https://oeis.org/A028560	https://oeis.org/A028566	https://oeis.org/A098603	http://oeis.org/A098847	http://oeis.org/A098848	http://oeis.org/A098849	http://oeis.org/A098850	http://oeis.org/A120071	OEIS / C

Figure C000776: Square-minus-squares Sequences

Examples:

- If $L = 2$, the diagonal sequence is $y^2 \pm 2y = (y \pm 1)^2 - 1$.
- If $L = 8$, the diagonal sequence is $y^2 \pm 8y = (y \pm 4)^2 - 16$.

The diagonal sequence is $y^2 \pm Ly = \left(y \pm \frac{L}{2}\right)^2 - \left(\frac{L}{2}\right)^2$. Equivalently, since L is even, writing $L = 2b$ gives $y^2 \pm 2by = (y \pm b)^2 - b^2$, with $b \in \mathbb{Z}$.

Quadratics	SNYPO	Description	OEIS
y(y ±0)	C000800	Integers in the form of square–0.	A000290
y(y ±2)	C000801	Integers in the form of square–1.	A005563
y(y ±4)	C000802	Integers in the form of square–4.	A028347
y(y ±6)	C000803	Integers in the form of square–9.	A028560
y(y ±8)	C000804	Integers in the form of square–16.	A028566
y(y ±10)	C000805	Integers in the form of square–25.	A098603
y(y ±12)	C000806	Integers in the form of square–36.	A098847
y(y ±14)	C000807	Integers in the form of square–49.	A098848
y(y ±16)	C000808	Integers in the form of square–64.	A098849
y(y ±18)	C000809	Integers in the form of square–81.	A098850
y(y ±20)	C000810	Integers in the form of square–100.	A120071

5. C000777-Oblong-minus-oblong sequences: Diagonal Offsets of Oblong Numbers

The figure C000777 presents the odd-offset analogue, given by

$$y(y \pm L) = y^2 \pm Ly$$

with L odd. When $L = 1$ this reduces to the two central diagonals of oblong numbers $y(y \pm 1)$. Each larger odd value of L generates two symmetric diagonals that can be expressed in terms of oblong differences. Explicitly,

$$y^2 + (2b - 1)y = (y + b)(y + b - 1) - b(b - 1)$$
$$y^2 - (2b - 1)y = (y - b)(y - b + 1) - b(b - 1)$$

Thus every odd-offset diagonal corresponds to 'oblong minus a fixed oblong,' with the fixed oblong given by $b(b - 1)$.

a	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	http://oeis.org/A000012
b	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	http://oeis.org/A000012
c	-110	-90	-72	-56	-42	-30	-20	-12	-6	-2	0	0	-2	-6	-12	-20	-30	-42	-56	-72	-90	http://oeis.org/A002378
	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	
10	-20	0	18	34	48	60	70	78	84	88	90	90	88	84	78	70	60	48	34	18	0	http://oeis.org/A132762
9	-38	-18	0	16	30	42	52	60	66	70	72	72	70	66	60	52	42	30	16	0	-18	http://oeis.org/A132761
8	-54	-34	-16	0	14	26	36	44	50	54	56	56	54	50	44	36	26	14	0	-16	-34	http://oeis.org/A132760
7	-68	-48	-30	-14	0	12	22	30	36	40	42	42	40	36	30	22	12	0	-14	-30	-48	http://oeis.org/A132759
6	-80	-60	-42	-26	-12	0	10	18	24	28	30	30	28	24	18	10	0	-12	-26	-42	-60	http://oeis.org/A119412
5	-90	-70	-52	-36	-22	-10	0	8	14	18	20	20	18	14	8	0	-10	-22	-36	-52	-70	http://oeis.org/A028569
4	-98	-78	-60	-44	-30	-18	-8	0	6	10	12	12	10	6	0	-8	-18	-30	-44	-60	-78	http://oeis.org/A028563
3	-104	-84	-66	-50	-36	-24	-14	-6	0	4	6	6	4	0	-6	-14	-24	-36	-50	-66	-84	http://oeis.org/A028557
2	-108	-88	-70	-54	-40	-28	-18	-10	-4	0	2	2	0	-4	-10	-18	-28	-40	-54	-70	-88	http://oeis.org/A028552
1	-110	-90	-72	-56	-42	-30	-20	-12	-6	-2	0	0	-2	-6	-12	-20	-30	-42	-56	-72	-90	http://oeis.org/A002378
0	-110	-90	-72	-56	-42	-30	-20	-12	-6	-2	0	0	-2	-6	-12	-20	-30	-42	-56	-72	-90	http://oeis.org/A002378
-1	-108	-88	-70	-54	-40	-28	-18	-10	-4	0	2	2	0	-4	-10	-18	-28	-40	-54	-70	-88	http://oeis.org/A028552
-2	-104	-84	-66	-50	-36	-24	-14	-6	0	4	6	6	4	0	-6	-14	-24	-36	-50	-66	-84	http://oeis.org/A028557
-3	-98	-78	-60	-44	-30	-18	-8	0	6	10	12	12	10	6	0	-8	-18	-30	-44	-60	-78	http://oeis.org/A028563
-4	-90	-70	-52	-36	-22	-10	0	8	14	18	20	20	18	14	8	0	-10	-22	-36	-52	-70	http://oeis.org/A028569
-5	-80	-60	-42	-26	-12	0	10	18	24	28	30	30	28	24	18	10	0	-12	-26	-42	-60	http://oeis.org/A119412
-6	-68	-48	-30	-14	0	12	22	30	36	40	42	42	40	36	30	22	12	0	-14	-30	-48	http://oeis.org/A132759
-7	-54	-34	-16	0	14	26	36	44	50	54	56	56	54	50	44	36	26	14	0	-16	-34	http://oeis.org/A132760
-8	-38	-18	0	16	30	42	52	60	66	70	72	72	70	66	60	52	42	30	16	0	-18	http://oeis.org/A132761
-9	-20	0	18	34	48	60	70	78	84	88	90	90	88	84	78	70	60	48	34	18	0	http://oeis.org/A132762
-10	0	20	38	54	68	80	90	98	104	108	110	110	108	104	98	90	80	68	54	38	20	http://oeis.org/A132763
	http://oeis.org/A132763	http://oeis.org/A132762	http://oeis.org/A132761	http://oeis.org/A132760	http://oeis.org/A132759	http://oeis.org/A119412	http://oeis.org/A028569	http://oeis.org/A028563	http://oeis.org/A028557	http://oeis.org/A028552	http://oeis.org/A002378	http://oeis.org/A002378	http://oeis.org/A028552	http://oeis.org/A028557	http://oeis.org/A028563	http://oeis.org/A028569	http://oeis.org/A119412	http://oeis.org/A132759	http://oeis.org/A132760	http://oeis.org/A132761	http://oeis.org/A132762	OEIS / C

Figure C000777 Oblong-minus-oblong Sequences

Examples:

If $L = 3$, the diagonal sequence is $y^2 \pm 3y$. Then,

- For the positive branch: $y^2 + 3y$ can be rewritten as $(y + 2)(y + 1) - 2$.
- For the negative branch: $y^2 - 3y$ can be rewritten as $(y - 2)(y - 1) - 2$.

- In both cases, the sequence is oblongs minus 2.

If $L = 9$, the diagonal sequence is $y^2 \pm 9y$.

- For the positive branch: $y^2 + 9y$ can be rewritten as $(y+5)(y+4) - 20$.
- For the negative branch: $y^2 - 9y$ can be rewritten as $(y-5)(y-4) - 20$.
- In both cases, the sequence is oblongs minus 20.

Generally, the diagonal sequence is $y^2 \pm Ly = \left(y \pm \frac{L+1}{2}\right)\left(y \pm \frac{L-1}{2}\right) - \frac{(L-1)(L+1)}{4} = \left(y \pm \frac{L+1}{2}\right)\left(y \pm \frac{L-1}{2}\right) - \frac{L^2-1}{4}$. Equivalently, since L is odd, writing $L = 2b+1$ gives $y^2 \pm (2b+1)y = \big(y \pm (b+1)\big)(y \pm b) - b(b+1)$, with $b \in \mathbb{Z}$.

Quadratics	SNYPO	Description	OEIS
y(y ±1)	C000900	Integers in the form of oblong–0.	A002378
y(y ±3)	C000901	Integers in the form of oblong–2.	A028552
y(y ±5)	C000902	Integers in the form of oblong–6.	A028557
y(y ±7)	C000903	Integers in the form of oblong–12.	A028563
y(y ±9)	C000904	Integers in the form of oblong–20.	A028569
y(y ±11)	C000905	Integers in the form of oblong–30.	A119412
y(y ±13)	C000906	Integers in the form of oblong–42.	A132759
y(y ±15)	C000907	Integers in the form of oblong–56.	A132760
y(y ±17)	C000908	Integers in the form of oblong–72.	A132761
y(y ±19)	C000909	Integers in the form of oblong–90.	A132762
y(y ±21)	C000910	Integers in the form of oblong–110.	A132763

5.1. Unified view

Both figures C000776 and C000777 fit the same general law $Y(y) = y(y \pm L)$. The only difference is whether L is even (square-minus-square families) or odd (oblong-minus-oblong families). In all cases the sequences are quadratic with leading coefficient 1, and the offset term $\pm Ly$ corresponds exactly to subtracting a fixed square or a fixed oblong. The case $L = 0$ gives the main diagonal of squares, and $L = 1$ gives the main diagonal of oblongs.

6. Mod 4 Constraints in Square and Oblong Differences

The families $y(y \pm L)$ display distinct residue behaviors modulo 4, depending on whether L is even or odd. This dichotomy corresponds to the square-minus-square and oblong-minus-oblong cases:

6.1. Square-minus-square family (L even)

Expressions of the form $y(y \pm 2b) = y^2 \pm 2by$ can be rewritten as $(y \pm b)^2 - b^2$, that is, “square minus square.Modulo 4, these sequences generate residues 0,1,3 but never 2. In particular:

- They include all odd integers ($\equiv 1$ or 3 mod 4).
- They also include all evens multiples of 4.
- They exclude the even numbers congruent to 2 mod 4.

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
9	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
8	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
7	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
6	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
5	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
4	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
3	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
2	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
0	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
-1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
-2	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
-3	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
-4	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
-5	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
-6	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
-7	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
-8	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
-9	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
-10	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0

Figure C003557: Square–minus–Square mod 4

6.2. Oblong-minus-oblong family (L odd)

Expressions of the form $y\big(y \pm (2b-1)\big) = y^2 \pm (2b-1)y$ can be rewritten as shifted differences of oblongs. Modulo 4, these sequences generate only residues 0 and 2. In particular:

- They include all the even integers ($\equiv 0 \; or \; 2 \; mod \; 4$).
- They exclude all odd integers ($\equiv 1 \; or \; 3 \; mod \; 4$).

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
9	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
8	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
7	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
6	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
5	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
4	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
3	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
1	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
-1	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
-2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
-3	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
-4	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
-5	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
-6	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
-7	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
-8	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2
-9	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0
-10	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0	0	2	2	0

Figure C003558: Oblong–minus–Oblong mod 4

6.3. Conclusion

Only integers congruent to $0\ mod\ 4$ can be represented in both families simultaneously. Every odd integer ($\equiv 1\ or\ 3$) must come from the square-minus-square side, while every even integer ($\equiv 2$) must come from the oblong-minus-oblong side.

7. C003559 Basic geometric Framework of the Multiplication Table: Circles, Hyperbolas, and Diagonal Symmetries

7.1. Starting with the Integer Grid

We begin the basic geometric construction of the multiplication table in the first quadrant of the Cartesian plane. The axes are marked not only with integers but also with their square roots, producing a unified framework that accommodates both rational and irrational coordinates.

All integer lattice points (x, y) are plotted as solid dots. Their products xy are labeled directly on the grid:

- Red for even values,
- Blue for odd values.

This parity-based coloring immediately reveals the interlacing of even and odd products across the grid.

In addition, the points $\left(\sqrt{n}, \sqrt{n}\right)$ are included as open circles, being the exact locations where the diagonal $x = y$ intersects the squares of natural numbers. These highlight the role of perfect squares as special lattice alignments that arise at both integer and non-integer coordinates.

This base figure establishes the foundation for next extensions of the multiplication table geometry. It shows how integer multiplication, parity, and square roots naturally coexist within the same plane, preparing the ground for circles, hyperbolas, and diagonal quadratic families.

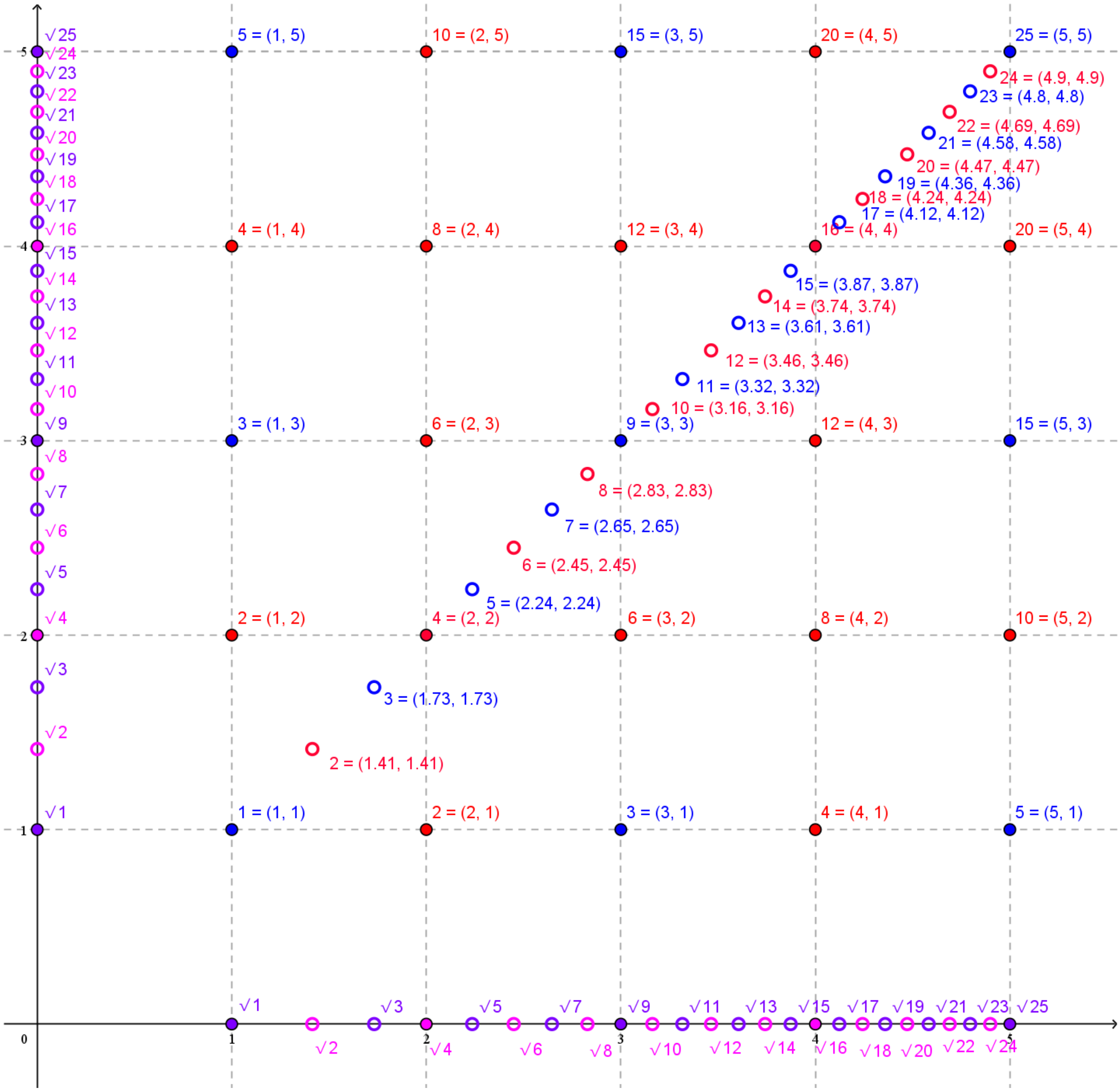


Figure C003560: MT Base Grid: Integer Points and Diagonal Roots

7.2. Extending the Grid with Root-Based Lines

In the second stage of construction, the multiplication grid is enriched by extending vertical and horizontal lines through every value marked on the axes. These include not only the positive integers but also the square roots of natural numbers.

The result is a rectangular mesh that combines rational and irrational coordinates in a single structure. Intersections occur at points such as $(2, \sqrt{5})$, $(\sqrt{7}, \sqrt{7})$, or $(\sqrt{9}, 4)$. When the square root corresponds to a perfect square, for example $\sqrt{9} = 3$, the intersection coincides with an integer lattice point, here (3,4).

This extension proves that the multiplication table is not confined to integer lattice points: it naturally generalizes into a dense analytic framework where products xy can be considered over both rational and irrational coordinates.

Thus, the grid anticipates the continuous families of hyperbolas $(xy = n)$ and circles $(x^2 + y^2 = n)$ to be introduced later, providing the structural background in which the discrete patterns of squares and oblongs will be embedded.

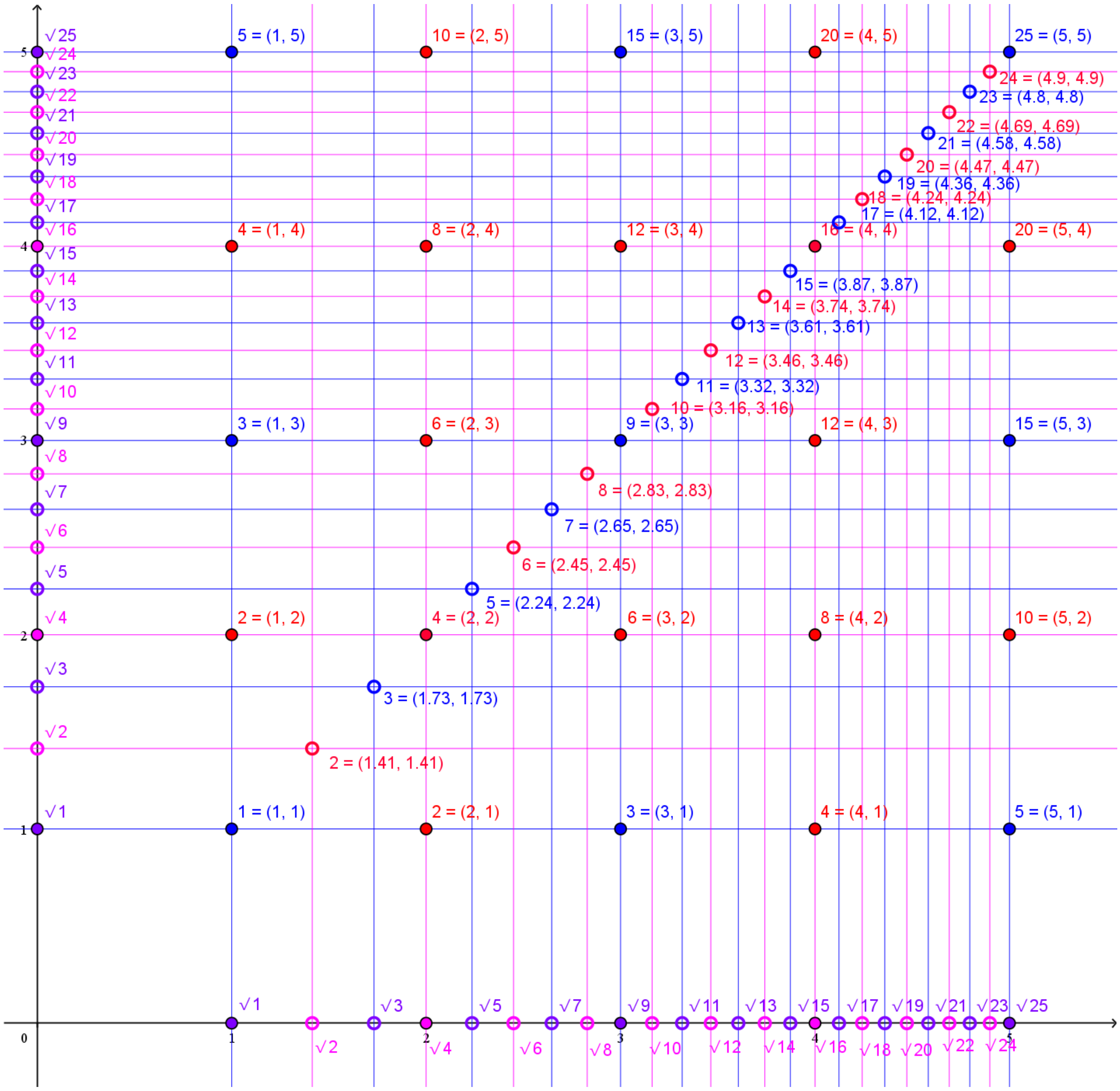


Figure C003561: MT Grid Extension: Vertical and Horizontal Lines

7.3. Adding Radial Circles: Parity in Distance from the Origin

In the third stage of the basic geometric construction, we enrich the multiplication table by adding concentric circles centered at the origin, defined by the equation $x^2 + y^2 = n$, where n is a positive integer.

Each circle groups all points at the same Euclidean distance $\sqrt{n}$ from the origin, introducing a radial layer of symmetry into the grid. To highlight arithmetic structure, we classify the circles by the parity of xy product:

- Red circles correspond to even values of xy.
- Blue circles correspond to odd values of xy.

This coloring emphasizes how integer lattice points distribute themselves according to the parity of their squared distance from the origin.

The result is a dual perspective: multiplication as a bilinear grid (products xy) and as a radial structure (sums of squares). These two views intersect in the lattice, producing visible layers of organization that unify arithmetic and geometry.

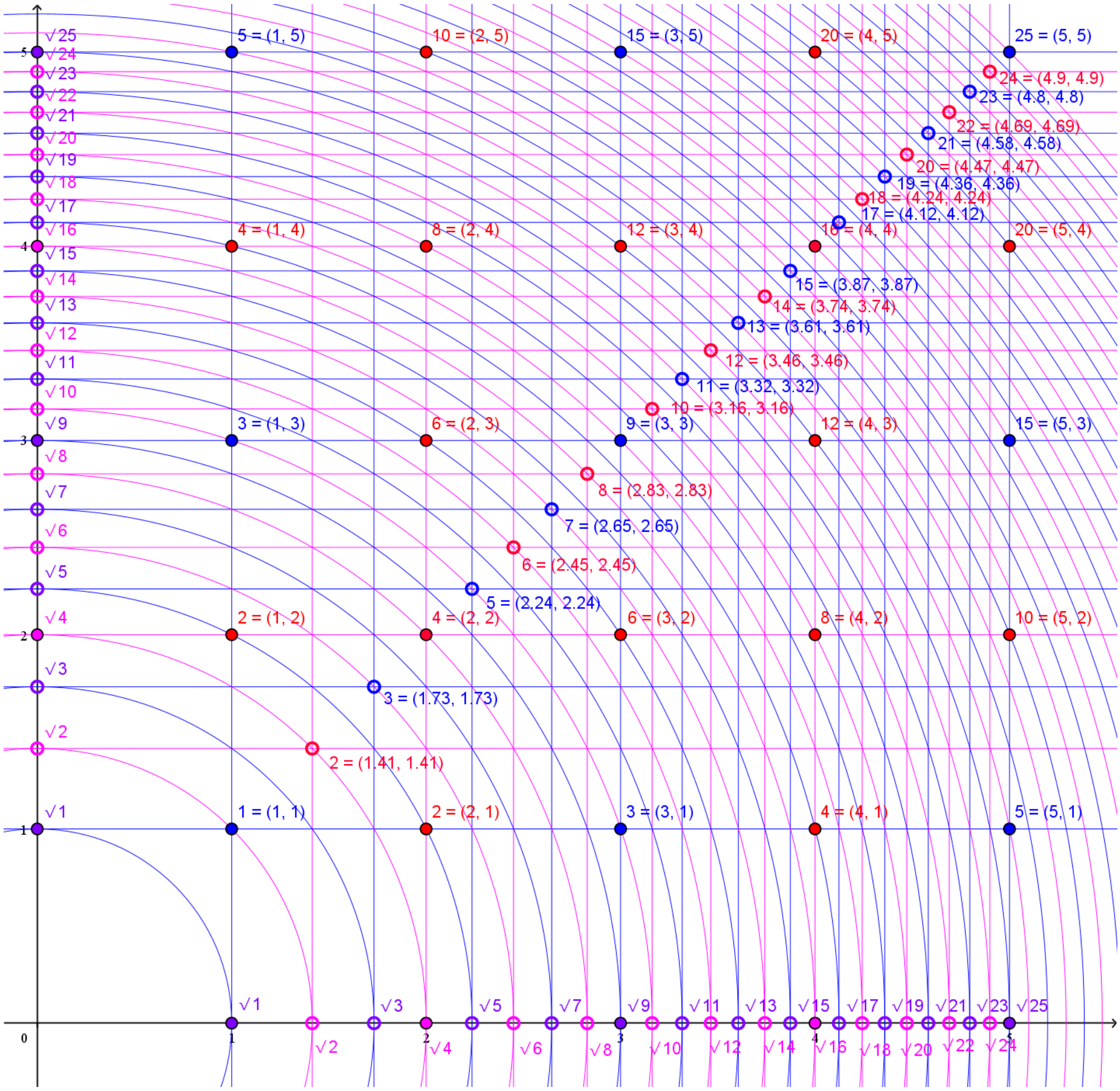


Figure C003562: MT Circles: Parity of Radius Squares ($x^2 + y^2 = n$)

7.4. Introducing Hyperbolas: Level Curves of Constant Product

In the fourth stage of construction, we enrich the multiplication grid by introducing hyperbolas defined by the equation $xy = n$. Each such curve is the locus of all points in the plane whose coordinates multiply to the same constant value n.

To emphasize both arithmetic and geometric structure, we distinguish two categories:

- Solid green hyperbolas correspond to integer values of n ($n = 1,2,3, \dots$).

- Dashed hyperbolas correspond to half-integer values of the form $n = \frac{2k-1}{2}$, or equivalently $n = m \pm 0.5$, where m is an integer.

Each hyperbola passes through infinitely many grid points, linking all factor pairs (x, y) with the same product. For example, the curve $xy = 12$ connects points such as $(3,4), (2,6)$, and $(\sqrt{12}, \sqrt{12})$. These curves extend the multiplication table into the continuous plane, unifying integer multiplication with its analytic background.

Several key features stand out:

- Symmetry: every hyperbola is symmetric with respect to the diagonal $x = y$, since $xy = n$ implies $yx = n$.
- Diagonal intersections: each curve intersects the line $x = y$ at $(\sqrt{n}, \sqrt{n})$, regardless of whether n is a perfect square.
- Density: as n increases, hyperbolas become more tightly packed, reflecting the multiplicative growth of products.

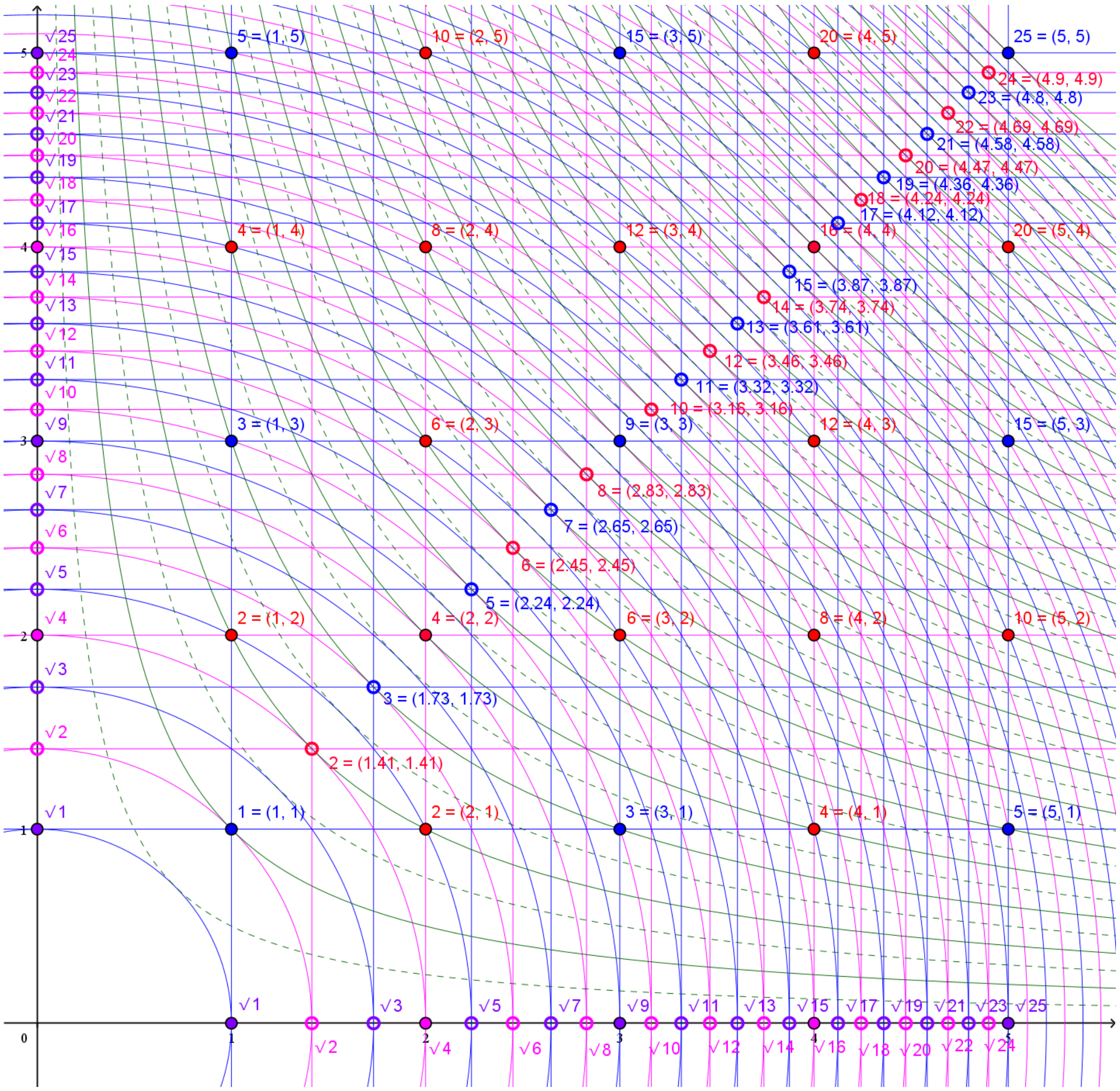


Figure C003563: MT Hyperbolas: Integer Products and Half-Integer Contours

Together, these hyperbolas reveal multiplication not only as discrete lattice points but also as a web of continuous geometric contours. They connect integer arithmetic to analytic geometry, showing how factor pairs align along smooth symmetric curves.

7.5. Adding Diagonal Quadratics at $\pm 45°$

In the final stage of construction, we complete the basic geometric framework of the multiplication table by overlaying diagonal lines inclined at $\pm 45°$. These diagonals correspond to families of quadratic sequences that systematically traverse the products generated by the table.

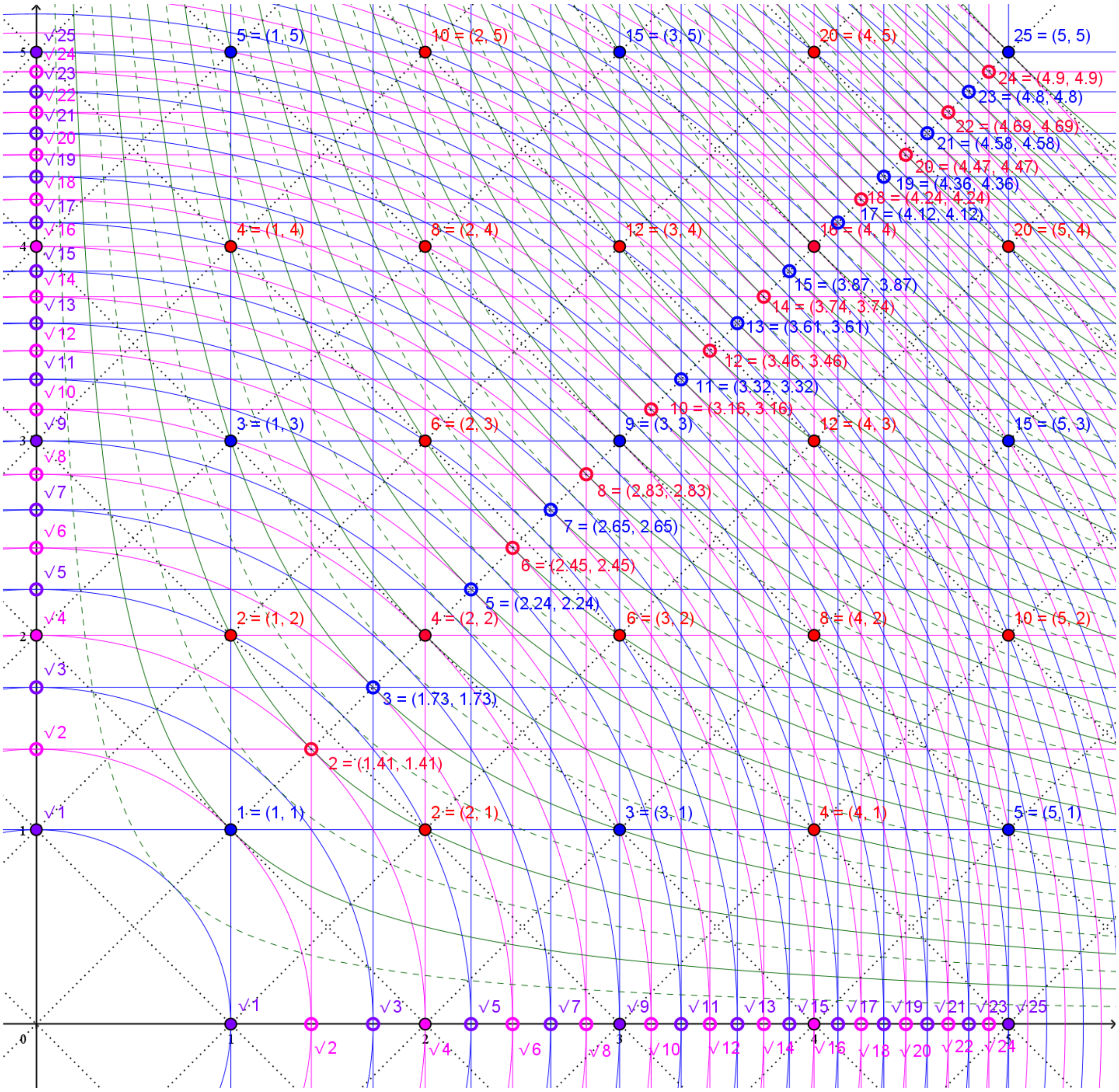


Figure C003564: Diagonal Families: $p^2 - q^2$ and $p(p-1) - q(q-1)$ over MT

Formally, they are expressed by equations of the type:

$$Y(y) = y(y \pm L)$$
$$X(x) = x(x \pm L)$$

where L is an integer shift parameter. These quadratic forms generate the two classical difference structures:

- Square-minus-square numbers, of the form $p^2 - q^2$.
- Oblong-minus-oblong numbers, of the form $p(p-1) - q(q-1)$.

Each diagonal captures a coherent progression of values that grow quadratically along symmetric paths. Unlike rows, columns, or circles, these diagonal families highlight how multiplication naturally encodes quadratic differences.

Integrated Framework

With the diagonals added, the full basic geometric structure of multiplication appears, combining:

- Radial circles ($x^2 + y^2 = n$), which encode parity through distance from the origin.
- Hyperbolas ($xy = n$), which trace loci of constant product.
- Root-based grids, which extend the lattice beyond integers by incorporating $\sqrt{n}$ coordinates.
- Integer-labeled products, which anchor the arithmetic within the visible grid.

This complete overlay reveals a unified interplay between arithmetic, algebra, and geometry. Squares, oblongs, and their difference sequences appear as natural symmetries woven into the multiplication process, all displayed within the first quadrant of the plane:

Beyond Quadratic Geometry

The basic geometric and algebraic patterns revealed by the multiplication table are rooted in quadratic structures, with circles, hyperbolas, and diagonal parabolas all deriving from second-degree polynomials. While these constructions already explain the vital role of squares, oblongs, and their difference sequences, they are only the foundational layer of arithmetic geometry. Further progress will require extending the framework to encompass higher-degree polynomials and broader geometric families. Cubic and quartic expressions, as well as more general algebraic curves, may reveal new layers of structure beyond the quadratic baseline. In this sense, the quadratic geometry of the multiplication table serves as the starting point.

7.6. The Quadratic Sequences of the Form $x = y(y \pm L)$

The structured arrangement of quadratic sequences, defined by $x = y(y \pm L)$, becomes especially clear when visualized through the $\pm 45°$ diagonal lines of the multiplication grid. These diagonals capture the two fundamental quadratic difference families:

- Yellow diagonals trace the sequences of the form $p^2 - q^2$ (square-minus-square numbers).
- Maroon diagonals trace the sequences of the form $p(p-1) - q(q-1)$ (oblong-minus-oblong numbers).

To clarify the classification of small integers under this framework:

- Number 1, sometimes termed the “oddest primeor “first non-composite positive integer, appears only in the square sequence $x = y(y+0)$ (OEIS A000290).
- Number 2, the next non-composite integer, the first prime, arises uniquely in the oblong sequence $x = y(y+1)$ (OEIS A002378).
- Number 3 belongs exclusively to $x = y(y+2)$, the square-minus-one sequence (OEIS A005563).
- Number 4, the first composite, belongs simultaneously to the square sequence $x = y(y+0)$ and to the oblong-minus-2 sequence $x = y(y+3)$ (OEIS A028552).
- Number 0 is present in every quadratic sequence of the form $x = y(y \pm L)$, a universal element often styled as “the oddest composite.”

This progression shows how primes and composites occupy distinct slots in the quadratic family framework. Far from being scattered, integers align along systematic diagonals governed by simple quadratic laws.

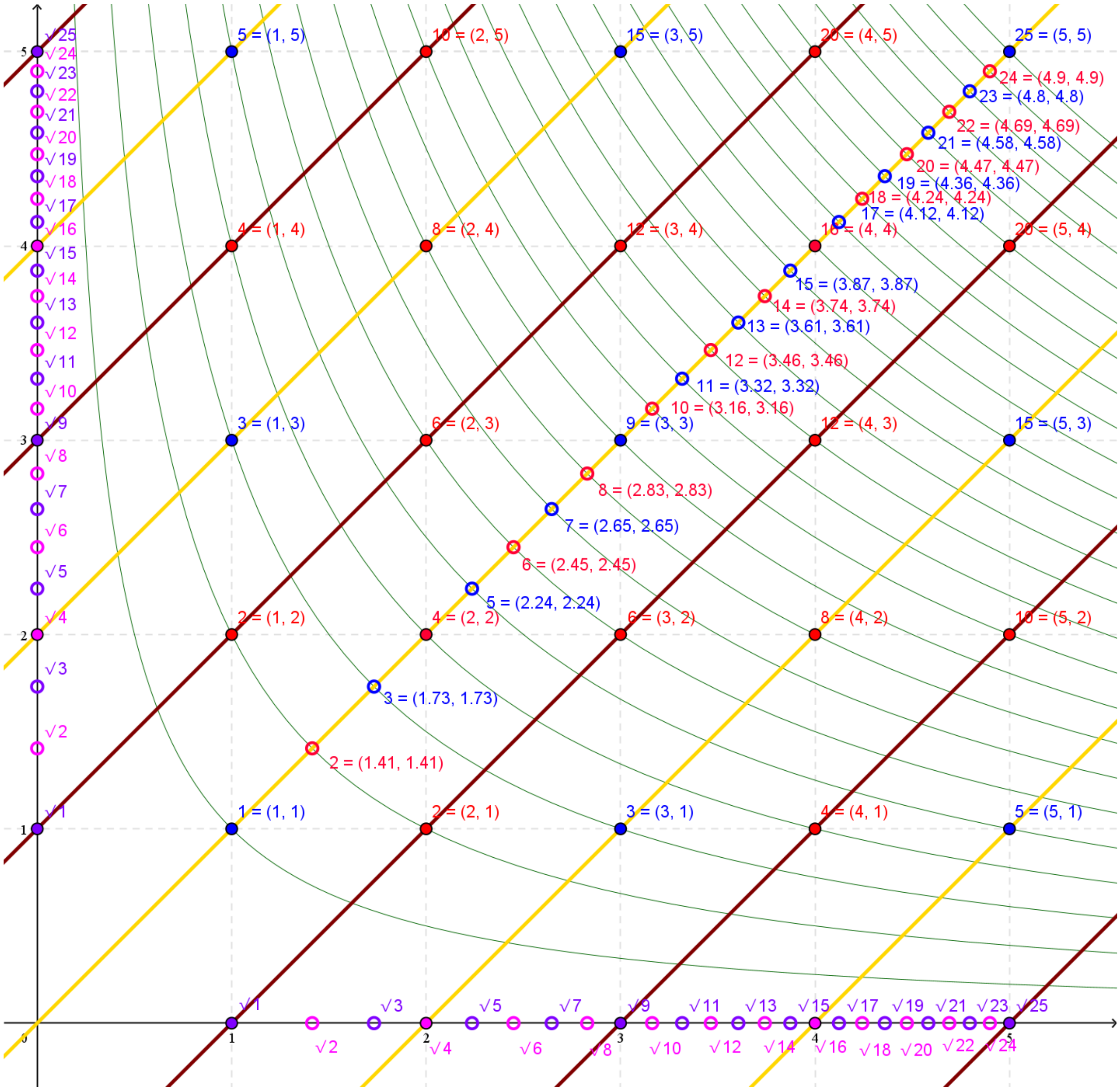


Figure C003565: Diagonal Quadratic Patterns: Square-Minus-Square (yellow) and Oblong-Minus-Oblong (maroon)

Figure C003565 illustrates these patterns, with yellow diagonals being $p^2 - q^2$ and maroon diagonals being $p(p-1) - q(q-1)$. Together with the hyperbolic and circular structures already introduced, they reveal how multiplication encodes a hidden lattice of quadratic regularities.

8. C000432 Triangular Multiplication Table (TMT): A Sufficient Framework

The study of prime distribution has long been linked to the geometry of divisors. For a fixed integer N, the divisor pairs (d_1, d_2) with $d_1 * d_2 = N$ lie on the hyperbola $xy = N$, and the split between small and large divisors is governed by the diagonal $x = y$, equivalently $\sqrt{N}$. In this sense, N is prime exactly when $xy = N$ has no integer points with $2 \le x \le \sqrt{N} \le y < N$ beyond the trivial pairs.

The TMT provides a natural extension of this divisor geometry. Instead of fixing a specific product N, one considers the entire multiplication grid xy, but restricted to the first octant $(x \ge 0, y \ge 0, y \le x)$. Within this reduced framework two distinguished families appear: perfect squares n^2 and oblong numbers $n(n \pm 1)$. Their geometric loci define the structural separation between composites and primes.

8.1. FMT, SMT, TMT

The multiplication table admits three natural restrictions:

- FMT (Full Multiplication Table): the entire Cartesian plane.
- SMT (Square Multiplication Table): the first quadrant $(x \ge 0, y \ge 0)$.
- TMT (Triangular Multiplication Table): the first octant $(x \ge 0, y \ge 0, y \le x)$.

Although the FMT spans the whole plane, it is fully symmetric around the origin and along the $\pm 45°$ diagonals. As a result, all key arithmetic, and geometric structures—squares, oblongs, diagonal quadratics, multiplication symmetries—are already present in the first octant.

This shows that the TMT alone is sufficient to capture the entire structure of the multiplication process. For this reason, all further developments in this work will focus on the TMT as the minimal yet complete representation of the multiplication table.

9										9 * 9 = 81
8									8 * 8 = 64	9 * 8 = 72
7								7 * 7 = 49	8 * 7 = 56	9 * 7 = 63
6							6 * 6 = 36	7 * 6 = 42	8 * 6 = 48	9 * 6 = 54
5						5 * 5 = 25	6 * 5 = 30	7 * 5 = 35	8 * 5 = 40	9 * 5 = 45
4					4 * 4 = 16	5 * 4 = 20	6 * 4 = 24	7 * 4 = 28	8 * 4 = 32	9 * 4 = 36
3				3 * 3 = 9	4 * 3 = 12	5 * 3 = 15	6 * 3 = 18	7 * 3 = 21	8 * 3 = 24	9 * 3 = 27
2			2 * 2 = 4	3 * 2 = 6	4 * 2 = 8	5 * 2 = 10	6 * 2 = 12	7 * 2 = 14	8 * 2 = 16	9 * 2 = 18
1		1 * 1 = 1	2 * 1 = 2	3 * 1 = 3	4 * 1 = 4	5 * 1 = 5	6 * 1 = 6	7 * 1 = 7	8 * 1 = 8	9 * 1 = 9
0	0 * 0 = 0	1 * 0 = 0	2 * 0 = 0	3 * 0 = 0	4 * 0 = 0	5 * 0 = 0	6 * 0 = 0	7 * 0 = 0	8 * 0 = 0	9 * 0 = 0
XY axes	0	1	2	3	4	5	6	7	8	9

Figure C000432_3: The map of the TMT. TMT – Multiplicative Expression Format

or

https://www.facebook.com/photo/?fbid=2117589555076444&set=gm.385407636449515&idorvanity=249087760081504

8.2. Historical Notes and OEIS References

The terminology TMT and SMT first appeared in the OEIS on Nov 08, 2001 (sequences A062858 A062859 A062856 and A062857). Earlier, A003991 (Mar 15, 1996) referred only to the multiplication table without distinguishing ranges. Many triangularly arranged sequences, such as A095833 and A075362, can be interpreted as TMT variants embedded in different octants.

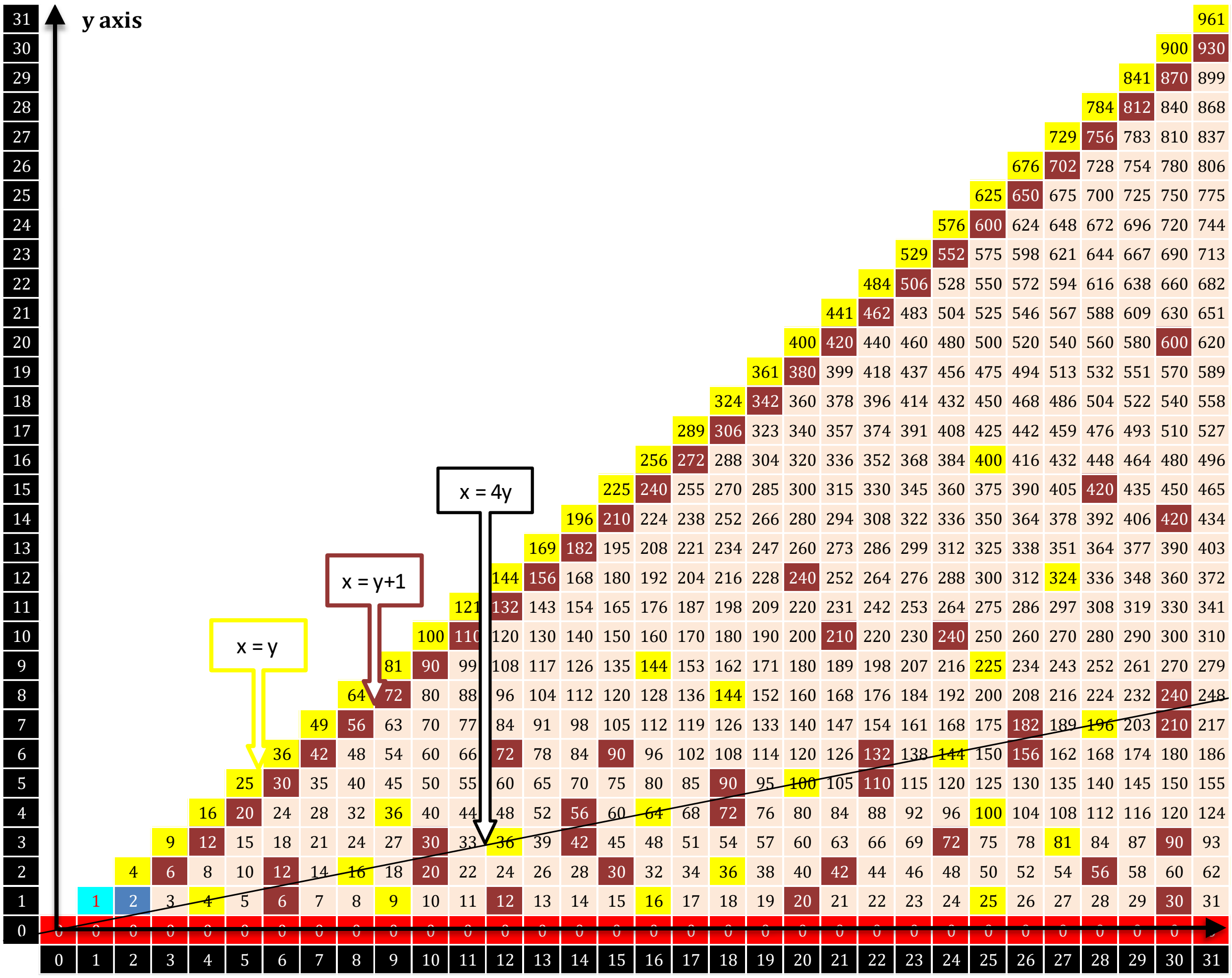


Figure C000432_2: 31×31 TMT in the first octant showing $x = 4y$ line in black. All 45° diagonal lines in the TMT are of the form $x = y + L$, where L is a constant. Each product in the lattice of the TMT is therefore of the form $x * y = y(y + L)$, representing the family of quadratic equations whose even and odd offsets (L even or L odd) generate, respectively, the square-minus-square and oblong-minus-oblong quadratic sequences described in Sections 4 (C000776) and 5 (C000777). Starting from the top square of each column, the same alternating pattern ($square - 0, oblong - 0, square - 1, oblong - 2, ...$) extends horizontally along the row through that square, forming the continuation of the column that would appear if the table were not truncated to the first octant. This reveals that the square/oblong interlacement is a two-dimensional symmetry inherited from the Full Multiplication Table (FMT) and preserved locally within the TMT. https://oeis.org/wiki/User:Charles_Kusniec/TMTinthe1stquadrant

8.3. Column–Top Progression and the Bidirectional Square–Minus–Square / Oblong–Minus–Oblong Interlacing in the TMT

Fix a column x in the TMT and read it from the top downward. Along that column, every lattice point lies on a 45° diagonal of the form $x = y + L$, where $L = x - y = 0,1,2,3, \dots$. Substituting this relation into the product gives the canonical quadratic expression $x * y = y(y + L)$, which decomposes all entries on the column into two complementary families governed solely by the parity of L.

Universal column and row interlacement:

Descending from the top $(y = x)$ of any column produces the deterministic interlacement:

$$square - 0$$
$$oblong - 0$$
$$square - 1$$
$$oblong - 2$$
$$square - 4$$
$$oblong - 6$$
$$square - 9$$
$$oblong - 12$$
$$\dots$$

The offset values increase by consecutive squares $(0,1,4,9,\dots)$ interlaced with consecutive oblong numbers $(0,2,6,12,\dots)$. Every product in the TMT (prime, composite, or unity) belongs to one of these two quadratic families (square-minus-square and oblong-minus-oblong) according to the parity of L.

The same interlacement also extends horizontally from the top square of each column: along the corresponding row that begins at the same square $(x = y)$, where $x^2 = y^2$, the sequence of products follows the identical alternation of square and oblong differences. Geometrically, these horizontal progressions are the natural continuation of the column beyond the triangular boundary. If the TMT were not truncated to the first octant, the vertical and horizontal interlacements would merge symmetrically, forming a continuous lattice of alternating square and oblong difference products across the full multiplication plane.

This bidirectional replication arises directly from the commutative property of multiplication $(x * y = y * x)$, which makes every column x a mirror of its corresponding row y. Because each product occupies both the vertical position (x fixed) and the horizontal position (y fixed) with the same value, the square-oblong alternation repeats symmetrically across both axes. If the TMT were not truncated to the first octant, these vertical and horizontal interlacements would merge perfectly, forming a continuous two-dimensional lattice of alternating square-minus-square and oblong-minus-oblong products across the entire multiplication plane.

Arithmetic consequences:

The mod 4 structure immediately follows:

- square-minus-square products (L even) occupy residue classes $0,1,3 \ mod\ 4$, and
- oblong-minus-oblong products (L odd) occupy $0,2 \ mod\ 4$.

Together they exhaust the positive integers in the TMT, with multiples of 4 shared by both families.

For prime sieve analyses, one commonly restricts the vertical range to $1 \le y \le x - 2$, excluding the trivial upper points ($square - 0$ at $y = x$ and $oblong - 0$ at $y = x - 1$). Within this admissible range the interlacing persists unchanged.

Illustrative examples:

Column $x = 11$:

$$L = 0 \ \rightarrow\ 11 * 11 = 121 = square - 0$$
$$L = 1 \ \rightarrow\ 11 * 10 = 110 = oblong - 0$$
$$L = 2 \ \rightarrow\ 11 * 9 = 99 = square - 1$$
$$L = 3 \ \rightarrow\ 11 * 8 = 88 = oblong - 2$$
$$L = 4 \ \rightarrow\ 11 * 7 = 77 = square - 4$$

And so forth.

Column x=10:

$$L = 0 \;\rightarrow\; 100 = square - 0$$
$$L = 1 \;\rightarrow\; 90 = oblong - 0$$
$$L = 2 \;\rightarrow\; 80 = square - 1$$
$$L = 3 \;\rightarrow\; 70 = oblong - 2$$

And so forth.

Interpretation:

This vertical interlacing is the column manifestation of the global quadratic law $x * y = y(y + L)$ described in Sections 4 (C000776) and 5 (C000777). Even L generates the square-minus-square sequences of the form $y(y + 2b) = (y + b)^2 - b^2$ of Section 4, while odd L produces the oblong-minus-oblong sequences $y^2 + (2b + 1)y = \big(y + (b + 1)\big)(y + b) - b(b + 1)$ of Section 5. Immediately below the top of every column lie the entries $square - 0$ and $oblong - 0$, after which the two families interleave with offsets b^2 and $b(b + 1)$. This systematic interlacing governs all products in the TMT.

8.4. Reduction to Odd Columns

Sufficiency can be pushed even further. Not only is the TMT enough to reproduce the full structure of the multiplication table, but in fact a single quarter of the TMT, namely the odd columns in the vertical range $1 \le y \le x - 2$, already encodes all essential information. In Section 11 (Lemma: Parabolic loci of squares and oblongs), we will tighten this admissible range to $1 \le y \le floor\left(\frac{x}{4}\right)$ and explain the geometric and modular reasons for that sharper bound; for the introductory discussion, $1 \le y \le x - 2$ is sufficient and keeps the picture simple.

In this reduced framework, the coverage by square and oblong families coincides exactly with the set of composites, while the uncovered columns coincide precisely with the primes on the positive integer line. This reduction explains why the TMT is not merely a convenience but a natural setting for sieve-like phenomena.

9. The Square Threshold Law for Complementary Divisors

9.1. Divisor Pairs in the TMT

Throughout this section, m denotes the product represented by a lattice point $(x, y) = (d_2, d_1)$ in the TMT , where $m = d_1 * d_2$ and $d_1 \le d_2$. The variable n refers to the parameter of the square or oblong numbers (n^2, $n(n \pm 1)$, etc.).

The restriction to the triangular range given by $y \le x$ encodes the pair of complementary divisors relation: d_1 is the smallest complementary divisor of m, while d_2 is the largest complementary divisor.

Because of symmetry, the complementary pair (d_2, d_1) corresponds to $(-d_1, -d_2)$ in one of the other octants, but within the TMT considered in this study, we retain only the positive representative.

9.2. Threshold Phenomenon

The diagonal $y = x$ plays a decisive role: for any product $xy = m$ strictly below n^2, the point (n, n) lies strictly above the hyperbola $xy = m$, so n cannot be realized as the smallest complementary divisor.

9.3. Theorem (Threshold for the Smallest Complementary Divisor)

Let $n \geq 1$. For any product $m = x * y$ with $0 < m < n^2$, the smallest complementary divisor satisfies $d_1 \leq \sqrt{m} < n$ hence $d_1 \leq n - 1$. Therefore, n never occurs as the smallest complementary divisor of any product $m < n^2$. The first product for which n appears as d_1 is precisely $m = n^2$, corresponding to the lattice point $(x, y) = (n, n)$.

Proof:

Suppose, for contradiction, that n is the smallest complementary divisor of some product $m < n^2$. Then $m = n * d_2$ for some integer $d_2 \geq n$ (because the smallest factor cannot exceed the largest). Hence $m \geq n * n = n^2$, contradicting the assumption $m < n^2$. Conversely, for $m = n^2$ the complementary pair is $(x, y) = (d_2, d_1) = (n, n)$, so n serves simultaneously as both the smallest and the largest complementary divisor at that point. This point marks the unique inversion of divisor roles: n, which for all smaller products acts solely as the largest complementary divisor, becomes from $m = n^2$ onward the smallest complementary divisor in every larger product.

9.4. TMT Interpretation

On the TMT lattice, this result shows a clear inversion boundary along the square diagonal $x = y$. The integer n first occurs as the smallest complementary divisor precisely at the lattice point $(x, y) = (n, n)$, corresponding to the square $m = n^2$. For every multiple $m = d_1 * n$ with $1 \leq d_1 < n$, the smallest complementary divisor along the column $x = n$ is d_1 (not n). In those cases, n appears only as the largest complementary divisor (d_2). At $m = n^2$, the roles invert: n, formerly the largest divisor, becomes thereafter the smallest complementary divisor for all larger products.

9.5. Corollaries

- If $d_1 * d_2 = m < n^2$ and $n|m$, then $n > d_1$ and in fact $n = d_2$ is the largest complementary divisor.
- If $d_1 * d_2 = m = n^2$, then $d_1 = d_2 = n$.
- If $d_1 * d_2 = m > n^2$ and $n|m$, then $n < \sqrt{m}$ and $n = d_1$ is the smallest complementary divisor.

9.6. Examples

For $n = 4$: below 16, the multiples 4,8,12 have smallest complementary divisors 1,2,3. Here 4 is always the largest complementary divisor. At $16 = 4^2$, the pair (4,4) appears, marking the threshold.

For $n = 5$: below 25, the multiples 5,10,15,20 have smallest complementary divisors 1,2,3,4. At $25 = 5^2$, the pair (5,5) appears.

9.7. Conceptual Significance

This shows a well-defined structural law: the integer n becomes visible as the smallest complementary divisor (d_1) precisely at its square n^2. In the geometry of the TMT, this explains why the square diagonal $x = y$ acts as the frontier line between two regimes: below n^2, n is too large to be the smallest factor, whereas at and beyond n^2, it functions as the smallest divisor for larger products.

This inversion of divisor roles across the diagonal $x = y$ encapsulates the duality between the upper and lower halves of the TMT: every largest divisor below the 45° diagonal becomes the smallest divisor above it, and vice versa.

10. Geometry and Coverage Criterion in the TMT

10.1. Basic Geometry of the TMT

The TMT is defined by the set of integer lattice points (x, y) with $1 \leq y \leq x$, from now on restricted to odd values of x. Each vertical column corresponds to a fixed odd x, and along that column one records the products xy. Within the admissible vertical range $1 \leq y \leq x - 2$ (nonempty for $x \geq 3$), these products capture the essential interactions between composite and prime structure.

In Section 11 (Lemma: Parabolic loci of squares and oblongs) we further tighten the range to $2 \leq y \leq \left\lfloor \frac{x}{4} \right\rfloor$ and explain the geometric and arithmetic rationale. The lower bound $y = 2$ is adopted because the base line $y = 1$ corresponds only to trivial products on square columns, where $x * 1 = x$, an odd square, yields exclusively the odd square composites themselves. For odd columns, even squares never appear, and the odd squares define the central columns that delimit the oblong sectors. Each sector is divided into four zones that play a vital role in Oppermann's conjecture [4]. Consequently, within these zones the uppermost odd-square column is never part of the search for primes, and values with $y = 1$ can be excluded without loss of generality.

Two distinguished vertical families of columns provide the natural framework for partitioning the table:

- Odd-square columns (yellow): $C_{odd^2}(m)$: $x = (2m - 1)^2$.
- (Even square)-minus-1 columns (orange): $C_{even^2-1}(m)$: $x = (2m)^2 - 1$.

Together with the oblong columns $C_{oblong}(m)$: $x = m(m - 1)$ (dashed even columns), these boundaries split the TMT into sectors of increasing size[4]. Each sector decomposes into four consecutive zones of equal width $2m + 1$, a feature that shows connection to Oppermann's conjecture.

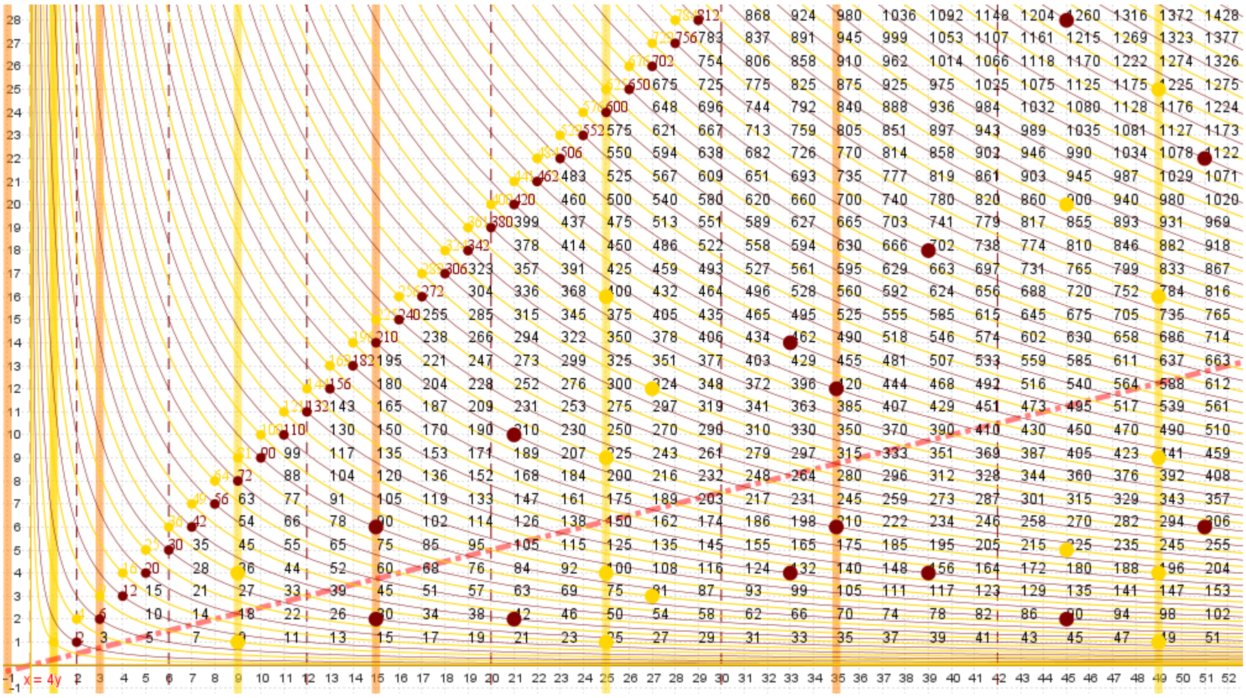


Figure C003588: Hyperbolic bundles of squares and oblongs over odd columns of the TMT.

Figure annotation: for each composite column $x \geq 4$, one may plot a lattice point (x, y) within $1 \leq y \leq x - 2$ obtained by rules above. If xy is square, color yellow and label (e.g., $9 * 4 = 36 = 6^2$); if xy is oblong, color maroon and label (e.g., $15 * 2 = 30 = 5 * 6$). For primes $p \geq 5$, explicitly mark "no solution in $1 \leq y \leq p - 2$ with py square or oblong.This matches the later theorem and visually separates composite (covered) from prime (uncovered) columns."

10.2. Square–Oblong Hyperbolic Coverage Criterion in the TMT

Throughout this section we restrict attention to the odd columns of the TMT. For a fixed odd column $x \geq 3$, we examine the nontrivial vertical range $1 \leq y \leq x - 2$.

A column x is said to be covered if there exists an integer y in that range such that xy is either a perfect square n^2 or an oblong $n(n \pm 1)$. The appearance of such a point (yellow for squares, maroon for oblongs) is coverage; columns that have no such points are uncovered. Later, we prove that all odd composite columns are covered and all odd primes are uncovered, setting up the square–oblong coverage sieve.

10.3. Notation and Visualization Conventions

We use distinct symbols for hyperbola families and for vertical column families:

- Square hyperbolas (yellow): $H_{square}(k)$: $xy = k^2$ with $k \in \mathbb{N}$.
- Oblong hyperbolas (maroon): $H_{oblong}(k)$: $xy = k(k-1)$ with $k \geq 2$.
- Oblong columns (maroon dashed): $C_{oblong}(m)$: $x = m(m-1)$.
- Odd-square columns (yellow): $C_{odd^2}(m)$: $x = (2m-1)^2$.
- (Even square)-minus-1 columns (orange): $C_{even^2-1}(m)$: $x = (2m)^2 - 1$.

A "yellow point is a lattice point (x, y) with xy a perfect square n^2.

A "maroon point is a lattice point (x, y) with xy an oblong number $n(n \pm 1)$.

Boundary convention: for $k = 1$ the oblong relation gives $xy = 0$, outside the domain $x \geq 1, y \geq 1$; hence oblong hyperbolas are taken with $k \geq 2$.

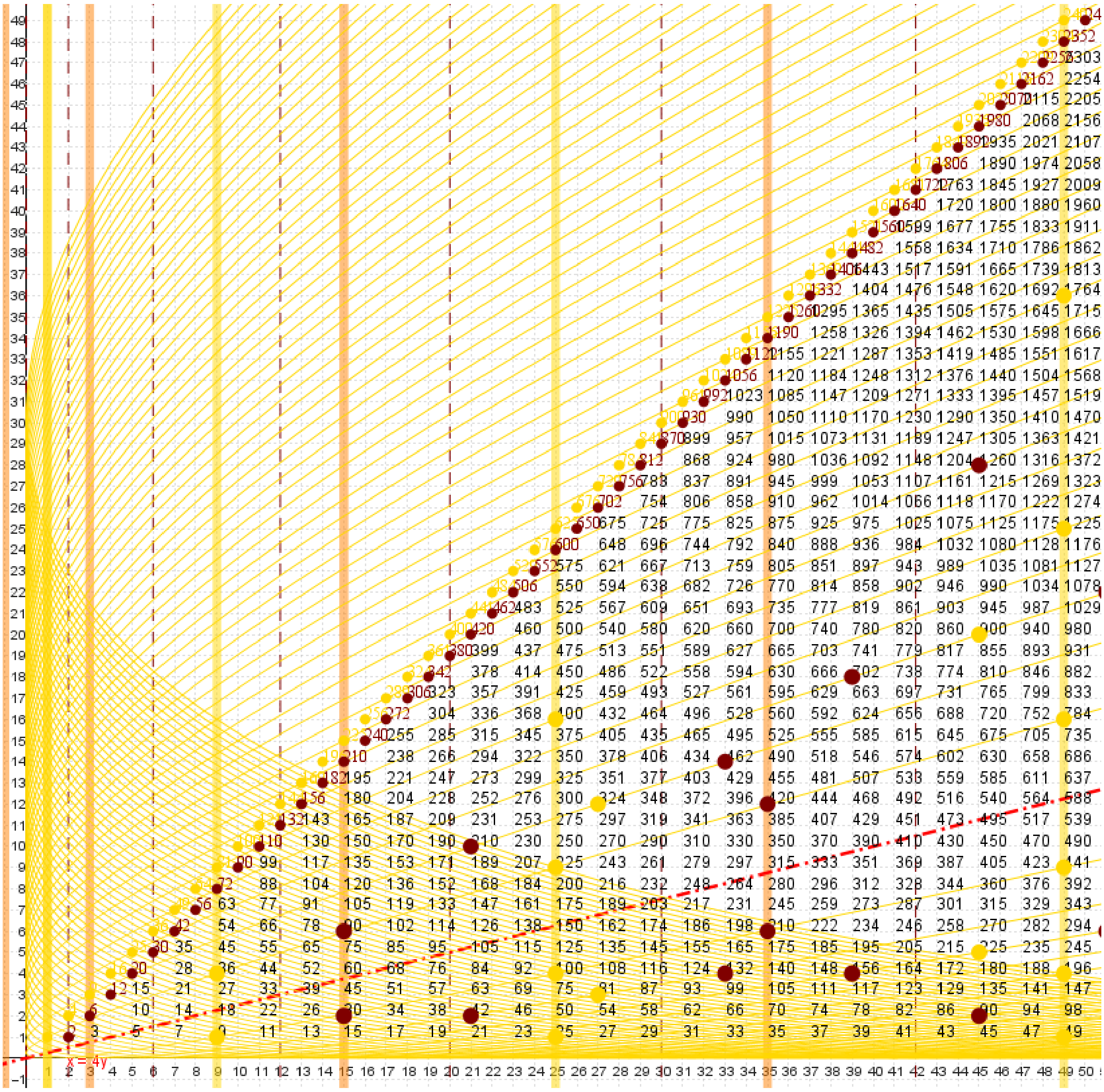
The square hyperbola family includes both even and odd squares, while the vertical columns $C_{odd^2}(m)$ mark only odd squares; this restriction is intentional.

11. Lemma: Parabolic Loci of Squares and Oblongs

A lattice point (x, y) in the TMT lies on a square-curve if and only if xy is a perfect square, and it lies on an oblong-curve if and only if xy is an oblong number $n(n \pm 1)$. Before introducing the implicit equations, we first outline the short geometric derivation that motivates both the odd parameter t and the shared axis of symmetry $x = 4y$.

11.1. Square Case:

We now precisely characterize the lattice points (x, y) in the TMT such that xy is a perfect square, showing that they form a one-parameter family of parabolas with a common axis of symmetry.



C003589 Parabolic loci of squares

Set-up

Assume $x, y, n \in \mathbb{Z}$ with $x \geq 1, y \geq 1, n \geq 1$ and $xy = n^2$ for some integer $n \geq 1$. We call (x, y) a square point.

Derivation (geometry-first)

- Define the parabola label (an odd integer) directly from the point:

$$t := x + 4y \pm 4n$$

Reason: this specific choice completes the square exactly and cuts the square root from the relation.

- Check the identity at once:

$$\left(t - (x + 4y)\right)^2 = (\pm 4n)^2 = 16n^2 = 16xy$$

Therefore every square point (x, y) lies on one of the curves:

$$\left(t - (x + 4y)\right)^2 = 16xy$$

for some odd t of the form $x + 4y \pm 4n$.

- Converse (so the description is exact, not just one-way):

If integers x, y, t satisfy $\left(t - (x + 4y)\right)^2 = 16xy$, then set $n := \frac{|t-(x+4y)|}{4}$. This n is an integer and $xy = n^2$. Hence the curve equation characterizes square points precisely.

What this form tells us:

- Common linear center and axis of symmetry:

The left side is a perfect square in t centered at $x + 4y$, so all such curves share the center $x + 4y$ and the axis of symmetry $x = 4y$. This axis serves as the vertical mirror that pairs lattice heights on each column.

- Explicit intersection heights on a fixed column x:

Rewrite the curve as a quadratic in y (expand the square and collect terms):

$$-x^2 + 8xy - 16y^2 + 2tx + 8ty = t^2$$

Solving this quadratic for y gives the two intersection heights:

$$-16y^2 + (8x + 8t)y + (-x^2 + 2tx - t^2) = 0$$

$$y_{\pm} = \frac{x + t \pm 2\sqrt{xt}}{4} = \left(\frac{\sqrt{x} \pm \sqrt{t}}{2}\right)^2$$

For $y_{\pm}$ to be integers we need $\sqrt{xt} \in \mathbb{Z}$ and the numerator divisible by 4; when $xy = n^2$ and $t = x + 4y \pm 4n$, we have $xt = (x \pm 2n)^2$, so $\sqrt{xt} = |x \pm 2n|$ and one root equals the original y.

- Pairing and "half-height consequence:

$$y_+ + y_- = \frac{x + t}{2}$$

Thus every square or oblong product below the symmetry axis $x = 4y$ has a corresponding partner above it on the same parabola. Practically, on a fixed column x you only need to scan $2 \leq y \leq \left\lfloor \frac{x}{4} \right\rfloor$; any crossing there implies a conjugate crossing above $\frac{x}{4}$ on that same curve.

- Parity of the label:

When x is odd (the regime used later for the sieve), $x + 4y$ is odd and $4n$ is even, so $t = x + 4y \pm 4n$ is automatically odd. Restricting t to odd values for all bundles preserves a consistent parity convention for both square and oblong families.

This parity constraint has a simple geometric consequence on the odd-column domain: since oblong numbers are necessarily even, intersections corresponding to $n(n \pm 1)$ appear only at even lattice heights along odd columns.

Note (Parity of Oblong Points):

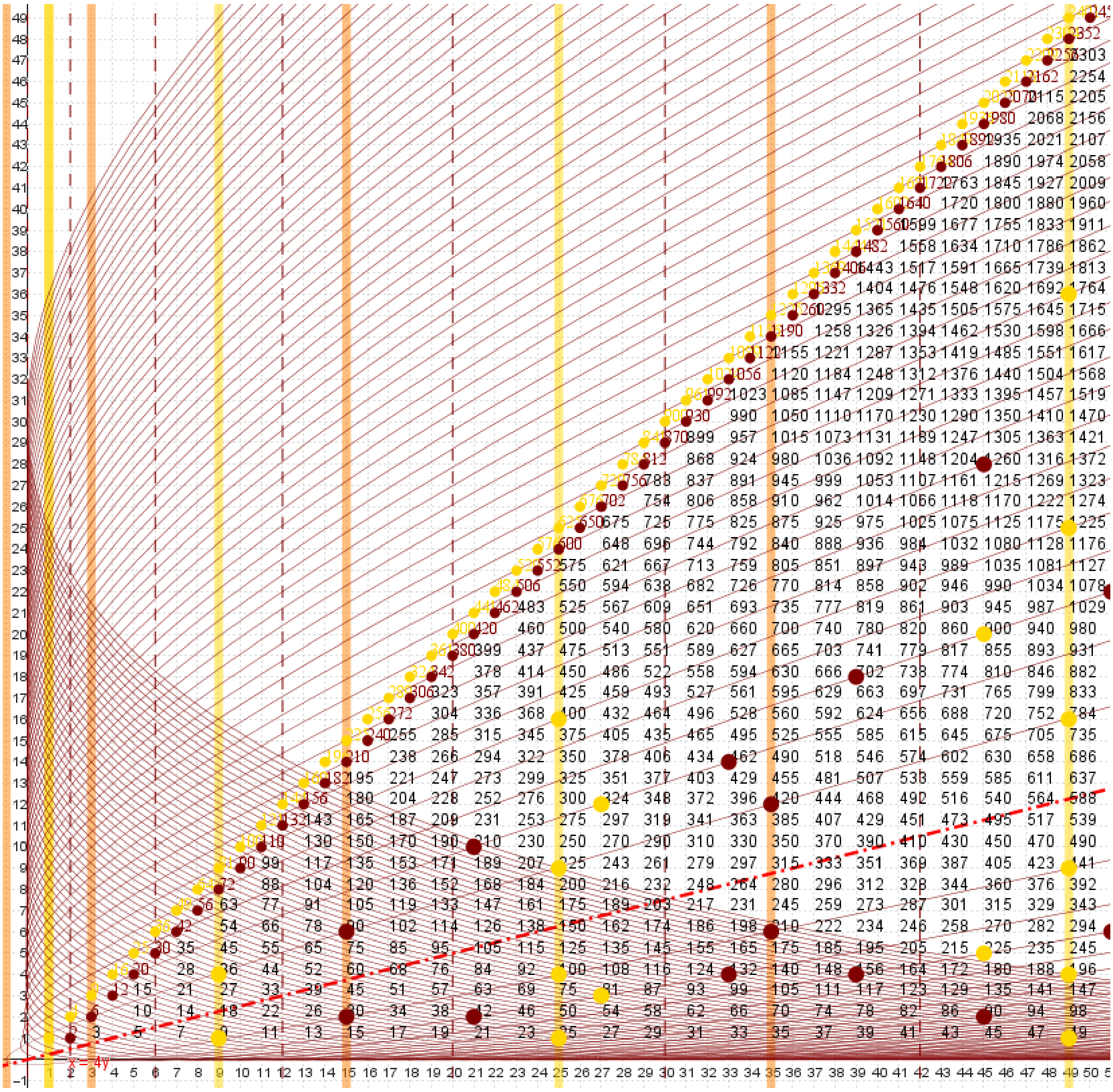
Since every oblong number $n(n \pm 1)$ is even, oblong intersections on odd columns of the TMT occur only at even heights y. Square intersections, by contrast, may appear at both parities.

Summary

Square points are exactly the integer solutions of $\left(t - (x + 4y)\right)^2 = 16xy$ with t odd. The center $x + 4y$ gives the common axis of symmetry $x = 4y$; the column intersections are $y_{\pm} = \left(\frac{\sqrt{x} \pm \sqrt{t}}{2}\right)^2$; and the pairing across $x = 4y$ explains the half-height scan $2 \leq y \leq \left\lfloor \frac{x}{4} \right\rfloor$ used throughout the coverage arguments.

11.2. Oblong Case

We next decide the lattice points (x, y) with $xy = n(n \pm 1)$, showing that they also lie on a one-parameter family of parabolas sharing the same axis of symmetry equivalently in the square case.



C003590 Parabolic loci of oblongs

Set-up

Let $x, y, n \in \mathbb{Z}$ with $x \geq 1, y \geq 1, n \geq 1$ and $xy = n(n \pm 1)$. We call (x, y) an oblong point.

The arithmetic identity that forces the 4

A number m is oblong if and only if $1 + 4m$ is an odd square:

$$m = n(n \pm 1) \Leftrightarrow \; 1 + 4m = 1 + 4n(n \pm 1) = (2n \pm 1)^2$$

This is the unique quadratic identity linking consecutive integers to odd squares, and it inherently introduces the factor 4.

Derivation (geometry-first, parallel to the square case)

The oblong condition $xy = n(n \pm 1)$ is equivalent to the odd-square condition $4xy + 1 = (2n \pm 1)^2$. This forces the "4that appears both in the axis $x = 4y$ and in the bundle equation below, and it fixes the parity convention t odd used throughout the oblong family.

Define the parabola label directly from the point:

$$t \coloneqq x + 4y \pm 2(2n \pm 1)$$

Reason: this formulation completes the square exactly and cuts the square root term.

- Check the identity at once:

$$\big(t - (x + 4y)\big)^2 = \big(\pm 2(2n \pm 1)\big)^2 = 4(2n \pm 1)^2 = 4(4xy + 1)$$

Therefore every oblong point (x, y) lies on one of the curves:

$$\big(t - (x + 4y)\big)^2 = 4(4xy + 1)$$

for some odd t of the form $x + 4y \pm 2(2n \pm 1)$.

- Converse (exact characterization)

If integers x, y, t satisfy $\big(t - (x + 4y)\big)^2 = 4(4xy + 1)$, then $4xy + 1 = t^2$ is an odd square, hence $xy = n(n \pm 1)$. Thus the curve equation characterizes oblong points precisely.

What this form tells us:

- Common linear center and axis of symmetry:

The left side is again a perfect square in t centered at $x + 4y$, so the oblong curves share the same center $x + 4y$ and the same axis of symmetry $x = 4y$ as the square family.

- Explicit intersection heights on a fixed column x:

Rewrite the curve as a quadratic in y and solve:

$$-x^2 + 8xy - 16y^2 + 2tx + 8ty = t^2 - 4$$

$$\Leftrightarrow$$

$$-16y^2 + (8x + 8t)y + (-x^2 + 2tx - t^2 + 4) = 0$$

So,

$$y_\pm = \frac{x + t \pm 2\sqrt{xt + 1}}{4}$$

Integrality requires $\sqrt{xt + 1} \in \mathbb{Z}$ and the numerator divisible by 4. For an oblong point with t chosen as $x + 4y \pm 2(2n \pm 1)$, see that:

$$xt + 1 = x\big(x + 4y \pm 2(2n \pm 1)\big) + 1 = x^2 + (4xy + 1) \pm 2x(2n \pm 1) = \big(x \pm (2n \pm 1)\big)^2$$

so $\sqrt{xt + 1} = |x \pm (2n \pm 1)|$ is an integer, and one root equals the original height y.

- Pairing and "half-height consequence:

As in the square case,

$$y_+ + y_- = \frac{x + t}{2}$$

Hence every intersection below the axis of symmetry $x = 4y$ has a corresponding partner above it on the same parabola. On a fixed odd column x it suffices to scan $2 \leq y \leq \left\lfloor \frac{x}{4} \right\rfloor$; any crossing there implies a conjugate crossing above $\frac{x}{4}$ on that curve.

- Parity of the label

When x is odd, $x + 4y$ is odd and $2(2n \pm 1)$ is even, so $t = x + 4y \pm 2(2n \pm 1)$ is automatically odd. Using odd t for both families keeps the formulations aligned.

Summary

Oblong points are exactly the integer solutions of $\left(t-(x+4y)\right)^2 = 4(4xy+1)$ with t odd. The center $x+4y$ yields the common axis $x=4y$, the column intersections are $y_{\pm} = \frac{x+t\pm2\sqrt{xt+1}}{4}$; and the same pairing across $x=4y$ supports the half-height scan $2 \le y \le \left\lfloor \frac{x}{4} \right\rfloor$ used in the coverage arguments.

11.3. Completion-Of-Square Alignment

Writing the square and oblong families in the same implicit form and completing the square in the parameter t produces a common linear center $x+4y$:

$$squares\text{: } \left(t-(x+4y)\right)^2 = 16xy,$$

$$oblongs\text{: } \left(t-(x+4y)\right)^2 = 4(4xy+1),$$

with t restricted to odd integers along each bundle. The axis $x=4y$ is therefore the shared axis of symmetry of both bundles; the same constant 4 that appears in $1+4m$ reappears here as the weight of y in $x+4y$.

11.4. Why The Axis Is $x=4y$ And Where the Constant 4 Comes From

There are two pictures of complementarity that are similar in spirit but arise from different mechanisms. For a fixed integer N, divisor pairs (d_1, d_2) with $d_1 * d_2 = N$ appear as integer points on the hyperbola $xy=N$; the small/large split is controlled by the diagonal $x=y$, equivalently by $\sqrt{N}$. In contrast, when we restrict the special products that are perfect squares or oblongs and let N vary, the relevant loci are two parabolic bundles. On each vertical line $x=$const, intersections from a fixed parabola occur in a symmetrical pair y_+, y_-, and the line of symmetry is $x=4y$ (equivalently $y=\frac{x+t}{4}$ for the appropriate parameter t). Thus the "middlethat separates the complementary heights is $x=y$ in the fixed-N divisor picture, but $x=4y$ in the square/oblong bundle picture.

The parabolic bundles of squares and oblongs within the TMT clearly show this underlying symmetry. On each vertical line $x=const$, intersections occur in symmetric pairs reflected across the axis $x=4y$, shown as a thick dotted line. This visual symmetry directly explains the operational half-height rule and provides the geometric foundation for Lemma 1 and Lemma 2 below.

11.5. Corollary (Visual Decomposition into Parabolic Bundles)

In the TMT the set of square points $\{(x,y)\text{: } xy=n^2\}$ and the set of oblong points $\{(x,y)\text{: } xy=n(n\pm1)\}$ each form a bundle of parabolas with common axis $x=4y$. The two bundles differ only by a constant shift in their defining equations, so that for large parameters they become almost coincident and appear visually parallel along the axis.

This explains why, in graphical plots such as Figure C003584, the yellow square points and the maroon oblong points are seen as two overlapping families of symmetric curves centered on the line $x=4y$.

11.6. Consequences For Geometry and Counting

Because the center is $x + 4y$, intersections on a fixed column x occur in symmetric pairs:

$$y_{\pm} = \frac{x + t \pm 2r}{4}$$

with $r^2 = 16xy$ (square bundle) or $r^2 = 4(4xy + 1)$ (oblong bundle). Hence $y_+ + y_- = \frac{x+t}{2}$, and every value below $x = 4y$ has a companion above $x = 4y$ on the same parabola.

In practice, this leads to the 'half-height' rule used throughout: to identify all integer crossings on a column, it is sufficient to scan $2 \le y \le \left\lfloor \frac{x}{4} \right\rfloor$; no information is lost above, since each point there has a paired counterpart below.

11.7. Refined Height Bounds and Their Limits

The half-height rule can be made precise as follows.

Both curve bundles, square and oblong, share the quadratic core $\left(t - (x + 4y)\right)^2 = 16xy$, where $t = 2v + 1$. For a fixed column $x \ge 1$ and a fixed parameter $t \ge 1$, the intersections of either bundle with that column are exactly the two values:

$$y_{\pm} = \frac{\left(\sqrt{x} \pm \sqrt{t}\right)^2}{4} = \frac{x + t \pm 2\sqrt{xt}}{4}$$

From this explicit formula one at once obtains the inequalities $y_- \le \min\left\{\frac{x}{4}, \frac{t}{4}\right\}$ and $y_+ \ge \max\left\{\frac{x}{4}, \frac{t}{4}\right\}$. Moreover, $y_+ + y_- = \frac{x+t}{2}$, so the two intersection heights are paired by the involution $y \mapsto \frac{x+t}{2} - y$ on the same curve for fixed t.

Whenever $t \le 4x$, the pair straddles the horizontal line $y = \frac{x}{4}$, that is $y_- \le \frac{x}{4} \le y_+$. Consequently, for every curve with $t \le 4x$ that contributes a lattice point in the range $2 \le y \le x - 2$, one of its intersections on the column x lies at or below $\frac{x}{4}$ and the other lies at or above $\frac{x}{4}$. In this regime, to count or certify coverage on column x it is sufficient to scan $2 \le y \le \left\lfloor \frac{x}{4} \right\rfloor$; any value in that range automatically has a conjugate value above $\frac{x}{4}$ on the same curve.

Curves with $t > 4x$ need not straddle $y = \frac{x}{4}$; both intersections may lie above $\frac{x}{4}$. Thus the "scan only up*to $\frac{x}{4}$reduction is valid for all contributions from parameters $t \le 4x$, but it is not an unconditional reduction for arbitrary t. In practical plots where t is truncated, for example $v \le V$ with $2V + 1 \le 4X_{max}$r the displayed data. For a complete proof, however, one must explicitly account for, or set up bounds on, the contribution from the $t > 4x$ range.

11.8. Connection With Decagonal Number Sequences

Along each odd column x of the TMT, the admissible range $2 \le y \le x - 2$ includes two extremal odd values that lie closest to the symmetry axis $x = 4y$. These extremal values always lie on one of the four neighboring diagonals $x = 4y \pm 1$ and $x = 4y \pm 3$, and the products generated at those intersections align precisely with classical decagonal number sequences.

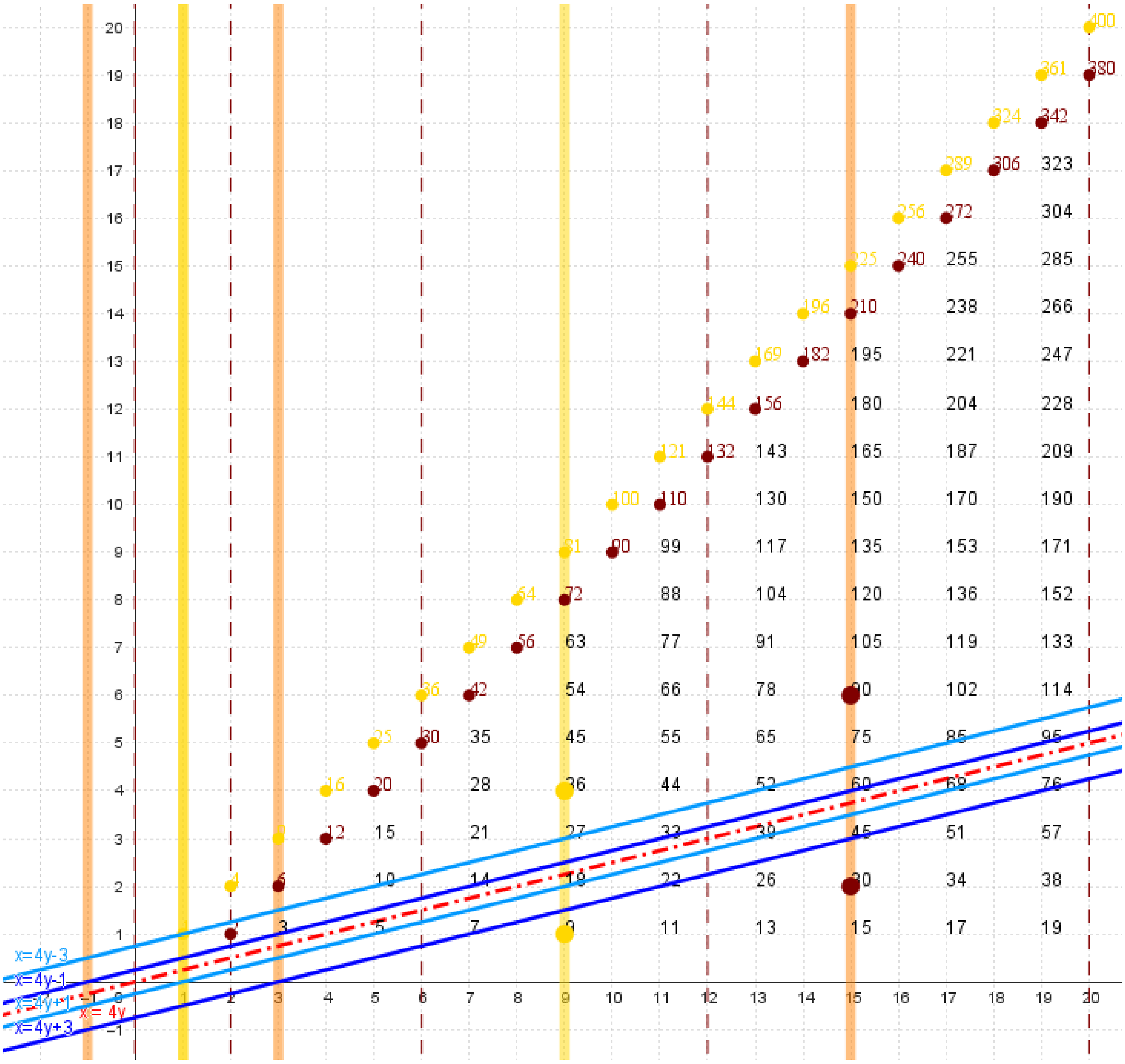


Figure C003586 Five diagonals of slope $1:4$ in the TMT.

The central dashed red line is the axis of symmetry $x = 4y$. The dark blue diagonals $x = 4y - 1$ and $x = 4y + 3$ intersect the odd columns $x \equiv 3 \ (mod\ 4)$. The light blue diagonals $x = 4y - 3$ and $x = 4y + 1$ intersect the odd columns $x \equiv 1 \ (mod\ 4)$. Extremal odd products at these intersections generate classical decagonal sequences.

The decagonal families arise naturally in the TMT from the extremal odd values nearest to the axis of symmetry $x = 4y$. The mechanism is geometric: each of the four diagonal lines of slope $1:4$ decides which odd columns are intercepted. Once y is fixed, the line equation provides x directly, and therefore the product $N = xy$. In this way, the lines $x = 4y \pm 1$ and $x = 4y \pm 3$ generate quadratic expressions in y whose values coincide with classical decagonal sequences.

Case 1. Odd columns $x \equiv 1 \ (mod\ 4)$, given by $x = 4y + 1$ (light blue)

- Lower neighbors are produced along the line $x = 4y + 1$: $N = xy = (4y + 1)y = 4y^2 + y = 3y^2 + (y^2 + y)$, generating the sequence A007742.
- Upper neighbors are produced along the line $x = 4y - 3$: $N = xy = (4y - 3)y = 4y^2 - 3y = y^2 + 3(y^2 - y)$, generating the sequence A001107.

Case 2. Odd columns $x \equiv 3 \ (mod\ 4)$, given by $x = 4y - 3$ (dark blue)

- Lower neighbors are produced along the line $x = 4y + 3$: $N = xy = (4y + 3)y = 4y^2 + 3y = y^2 + 3(y^2 + y)$, generating the sequence A033954.
- Upper neighbors are produced along the line $x = 4y - 1$: $N = xy = (4y - 1)y = 4y^2 - y = 3y^2 + (y^2 - y)$, generating the sequence A033991.

In all four formulas above, quadratic expressions can be rewritten in terms of squares and oblongs; for example, $4y^2 + y = 3y^2 + (y^2 + y)$, that is, a decomposition into three square terms plus one oblong term. This structural decomposition is not incidental: it precisely reflects the square–oblong framework that underlies the sieve mechanism of the TMT, where primes appear as those columns that escape both families.

Conclusion.

In both congruence classes of odd columns, the extremal odd products nearest to the axis $x = 4y$ fall into the decagonal sequences A007742 and A033954, together with their complementary companions A001107 and A033991. The axis $x = 4y$ serves as the central organizing symmetry, while the neighboring lines $x = 4y \pm 1$ and $x = 4y \pm 3$ are the loci that generate the decagonal families.

11.9. What The Analogy Does and Does Not Mean

The shared "complementary pairs intuition is valid: in both settings each admissible point comes with a partner reflected across a reference line. However, the reference lines differ because the mechanisms differ. For divisors of a fixed product N the natural symmetry is $x = y$, equivalently $\sqrt{N}$. For the special-product bundles (squares and oblongs, with N varying along the curves) the natural symmetry is $x = 4y$, dictated simultaneously by the identity $1 + 4m = (odd\ square)$ and by the algebraic completion-of-square in t. The constant 4 is therefore not a modified $\sqrt{x}$, but the exact coefficient linking consecutive integers to odd squares and defining the linear center $x + 4y$ of the parabolic bundles. This structural constant 4 thus governs both the geometric symmetry and the arithmetic decomposition central to the TMT framework.

12. Squares in Central Columns and Oblongs along Sector Walls in the TMT

12.1. Lemma 1 (In the Range $2 \leq y \leq x - 2$, Every Center Column Admits Square Values, but No Oblongs)

Fix an odd square column $x = a^2$ with $a \geq 3$. Within the range $2 \leq y \leq x - 2$, there is no integer y for which $x * y$ is oblong; all admissible heights on the column $x = a^2$ correspond exclusively to square products.

Proof: If $x * y$ is oblong then $1 + 4 * x * y = r^2$ with r odd. With $x = a^2$ we have $4a^2 \mid (r - 1)(r + 1)$. Since both a and r are odd, $\gcd(r - 1, r + 1) = 2$ so a^2 must divide exactly one of $r \pm 1$. Hence, we may set $r = a^2 t \pm 1$ for some integer t. Then

$$y = \frac{r^2 - 1}{4a^2} = \frac{a^2 t^2 \pm 2t}{4}$$

For y to be an integer, t must be even. Say $t = 2u$ with $u \geq 1$. Hence $y = a^2u^2 \pm u$. The smallest possibility is $u = 1$, which gives $y = a^2 - 1$ or $y = a^2 + 1$. Substituting $x = a^2$, this means $y = x - 1$ or $y = x + 1$. Both lie outside the range, since $x - 1 > x - 2$ and $x + 1 > x$. Therefore no oblong value occurs for $2 \leq y \leq x - 2$, while square values $y = d^2$ (with $4 \leq d^2 \leq x - 2$) do occur, always in symmetric pairs.

12.2. Lemma 2 (In the Range $2 \leq y \leq x - 2$, Every Sector Wall Admits At Least a Symmetric Pair of Oblong Values and May Also Admit Squares)

Let $m \geq 2$ and define the sector-wall column as $x = (2m)^2 - 1 = 4m^2 - 1$. Then at least the two heights:

$$y_1 = m(m-1)$$
$$y_2 = m(m+1)$$

lie in the range $2 \leq y \leq x - 2$ and satisfy:

$$x * y_1 = n_1(n_1 - 1)\ with\ n_1 = m(2m-1),$$
$$x * y_2 = n_2(n_2 - 1)\ with\ n_2 = m(2m+1).$$

Thus every sector wall column $x = 4m^2 - 1$ carries at least these two symmetric oblong values $y = m(m \pm 1)$.

Proof: Factor $x = (2m-1)(2m+1)$. Choose $n_1 = m(2m-1)$, yielding:

$$n_1(n_1 - 1) = m(2m-1)(m(2m-1) - 1) = (2m-1)(2m+1) * m(m-1)$$
$$= x * m(m-1) = x * y_1$$

Similarly, with $n_2 = m(2m+1)$ we get $n_2(n_2 - 1) = x * m(m+1) = x * y_2$.

For $m \geq 2$ we have $2 \leq m(m-1)$ and $m(m+1) \leq 4m^2 - 3 = x - 2$, ensuring that both heights lie within the admissible range.

12.3. Lemma 3 (In the Range $2 \leq y \leq x - 2$, Prime Columns Admit No Square or Oblong Values)

Let p be a prime with $p \geq 5$. Within the range $2 \leq y \leq p - 2$, there exists no integer y such that $p * y$ equals either a perfect square n^2 or an oblong number $n(n \pm 1)$.

Equivalently, along a vertical prime column $x = p$, the hyperbolas $xy = v^2$ and $xy = v(v-1)$ have no integer lattice points within the range $2 \leq y \leq p - 2$.

Proof: First suppose $p * y = n^2$. Then $p|n^2$, hence $p|n$; write $n = p * t$ with $t \in \mathbb{Z}$. It follows that $y = \frac{n^2}{p} = p * t^2 \geq p$, contradicting $y \leq p - 2$.

Next suppose $p * y = n(n \pm 1)$.

Since $\gcd(n, n \pm 1) = 1$, the prime p must divide exactly one of the two coprime factors.

- If $p|n$, then $y = \left(\frac{n}{p}\right)(n \pm 1) \geq 1 * (p-1) = p - 1$.
- If $p|(n \pm 1)$, then $y = n\left(\frac{n+1}{p}\right) \geq (p-1) * 1 = p - 1$.

In either case $y \geq p - 1$, again contradicting $y \leq p - 2$.

Therefore no such y exists within the range, and the nonexistence of integer crossings follows directly.

13. Prime Range Theorem: Geometric Structure of Prime Columns

13.1. Theorem

Fix an odd prime $p \geq 3$ and consider its corresponding vertical column$x = p$ in the TMT. The admissible range $1 \leq y \leq p$ decomposes into three distinct layers along the column:

(1) **Base point $(y = 1)$:** the product $p * 1 = p$ is the prime itself, which is neither square nor an oblong.

(2) **Interior range $2 \leq y \leq p - 2$:** every entry $p * y$ in this range is composite, yet neither a perfect square nor an oblong number.

(3) **Upper edge $y = p - 1$:** exactly one oblong appears at this height, namely $p * (p - 1) = n(n - 1)$ with $n = p$. This is the first oblong product on the prime column.

(4) **Top point $y = p$:** exactly one square appears at this height, namely $p * p = p^2$. This is the first square product on the prime column.

Example. For $p = 11$, the interior range $2 \leq y \leq 9$ yields the products 22,33,44,55,66,77,88,99. None are squares or oblongs. At $y = 10$ we have $110 = 11 * 10$ an oblong number of the form $n(n - 1)$ with $n = 11$. At $y = 11$ we have $121 = 11^2$ a square number.

Proof:

Suppose $p * y$ is a perfect square with $2 \leq y \leq p - 2$. Then p divides m since $m^2 = p * y$, hence $m = p * t$ and $y = p * t^2 \geq p$, contradicting $y \leq p - 2$. Thus no square occurs in the interior.

Now suppose $p * y = n(n \pm 1)$ with $2 \leq y \leq p - 2$. Then p divides exactly one of n or $n + 1$ implying $n(n \pm 1) \geq p(p - 1)$. Yet $p * y \leq p(p - 2) < p(p - 1)$, a contradiction. Thus no oblong occurs in the interior. At $y = p - 1$ we obtain $p * (p - 1) = n * (n - 1)$ with $n = p$, and at $y = p$ we obtain p^2. No other square or oblong product exist in the TMT column.

13.2. Geometric Interpretation

The three lines $b^2x - a^2y = 0$ and $b^2x - a^2y = \pm ab$ originate from the hyperbolic loci $xy = n^2$ (squares) and $xy = n(n \pm 1)$ (oblongs), which intersect the 45° diagonals at $x = y$ (for squares) and at $x = y + 1$ (for oblongs) within the TMT.

These equations are the linear projections of those hyperbolas:

- The central line $b^2x - a^2y = 0$ captures the intersections of the square family $xy = n^2$; and
- The two offset lines $b^2x - a^2y = \pm ab$ capture the intersections of the oblong family $xy = n(n \pm 1)$.

For each fixed integer pair (a, b), the central line represents all lattice points (x, y) whose products xy are perfect squares, while the offsets correspond to products lying one step above or below that square, namely the oblong numbers $n(n \pm 1)$.

Together, these three parallel lines are the "square–oblong product bundle along the 45° diagonals, providing a linear framework that reveals where the square and oblong hyperbolic families intersect.

Along the prime column $x = p$, the central line intersects only at $y = p$, and the two oblong offsets intersect only at $y \geq p - 1$, first intersection at $y = p - 1$.

Consequently, the entire interior range $2 \le y \le p - 2$ stays untouched by all members of these diagonal bundles.

13.3. Conceptual Consequence

Odd primes correspond exactly to those columns whose interior range is free of both square and oblong products. This observation justifies the strict cutoff at $y \le x - 2$ in the coverage definition: if the range were extended to $y = x - 1$ or $y = x$, every prime column would include an oblong or square at its top edge, thereby breaking the sieve classification.

14. Square/Oblong Coverage Criterion on a Column: Composites Are Covered, Odd Primes Are Not (by Lemma 3)

We work in the factor plane of the multiplication table, where each lattice point (x, y) is the product xy. Fix a column $x \ge 2$ and restrict attention to the bounded height range $2 \le y \le x - 2$.

A column x is said to be covered in this range if there exists an integer y with $2 \le y \le x - 2$ and coprime integers $a, b \ge 1$ such that (x, y) lies on one of the lines $b^2x - a^2y = 0$ or $b^2x - a^2y = \pm a * b$. Equivalently, x is covered in this range if and only if there exists y with $2 \le y \le x - 2$ for which xy is either a perfect square n^2 or an oblong $n(n \pm 1)$. (For $x = 2$ or $x = 3$ the range is empty.)

14.1. Corollary (Odd Prime Columns Are Not Covered; See Lemma 3)

For every odd prime $p \ge 5$, the column $x = p$ has no square or oblong value in the range $2 \le y \le p - 2$. This is exactly Lemma 3. The only remaining boundary cases are $x = 2$ and $x = 3$ (empty range) and $x = 4$, the unique composite with no values of either type within the admissible range.

14.2. Theorem (Composite Columns Are Covered, Except For $x = 4$)

Let $x \ge 2$ be composite. If $x \ne 4$, then there exists an integer y with $2 \le y \le x - 2$ such that xy is a perfect square or an oblong; equivalently, the column x is met in range by one of the line families $b^2x - a^2y \in \{0, \pm ab\}$.

Proof:

We divide the argument into two cases:

Case A (x has a nontrivial square factor):

Write $x = a^2m$ with integers $a \ge 2$ and $m \ge 1$.

- If $m \ge 2$, choose $y = m$. With $b = 1$ we have $b^2x - a^2y = x - a^2m = 0$, so (x, y) lies on a square line and $xy = (am)^2$ is a square. Bounds: since $y = m \ge 2$ and $x - y = m(a^2 - 1) \ge 2 * 3 > 2$, we obtain $2 \le y \le x - 2$.
- If $m = 1$, then $x = a^2$. For $a \ge 3$ set $y = 4$. With $b = 2$ we get $4a^2 - a^2 * 4 = 0$, so $xy = 4a^2 = (2a)^2$ and $4 \le x - 2$ since $a \ge 3$ $(x \ge 9)$. The remaining case $x = 4$ $(a = 2)$ is the unique composite not covered: its only admissible height $y = 2$ yields $4 * 2 = 8$ is neither square nor oblong.

Case B (x is squarefree):

B1 (x odd and squarefree):

Write $x = a * c$ where a is the smallest prime factor of x, so $3 \le a < c$ and $\gcd(a, c) = 1$. Let b_0 be the smallest positive solution of $b * c \equiv 1 \ (mod\ a)$. If $b_0 \le \frac{a}{2}$ use the "$+ab$ line; otherwise set $b := a - b_0$ and use the "$-ab$ line (then $b * c \equiv -1 \ (mod\ a)$). In either branch, set $b = \min\{b_0, a - b_0\}$ so $1 \le b \le \frac{a}{2}$. Intersecting with $x = a * c$ gives the integer height:

$$y = \frac{b^2 c \mp b}{a}$$

since $a|(b\text{^}2\ c \mp b) = b(bc \mp 1)$. By the standard parameterization, the table entry at that height is oblong: $xy = (ak)\big((ak) \pm 1\big)$ with $k = \frac{bc \mp 1}{a}$.

These bounds ensure that the height lies within the admissible range for all odd $x \ge 15$:

$y \le \frac{b^2 c + b}{a} \le \frac{x}{4} + \frac{1}{2} \le x - 2$, and $y \ge 2$ (if $b \ge 2$ it is immediate, if $b = 1$ we are in the "$-ab$branch and $y = (c + 1)$, $a \ge 2$).

Small odd squarefree cases below $x = 15$ are verified directly (for example, $x = 15 \rightarrow y = 2$; $x = 21 \rightarrow y = 10$; $x = 33 \rightarrow y = 4$).

B2 (x even and squarefree):

Write $x = 2s$ with odd $s \ge 3$.

If $s \ge 5$, take $n := s$ and $y := \frac{n(n-1)}{x} = \frac{s-1}{2}$, which is an integer satisfying $2 \le y \le x - 2$. If $s = 3$ (that is, $x = 6$), take $n := 4$ and $y = \frac{12}{6} = 2$. In both subcases $xy = n(n - 1)$ is oblong, hence (x, y) lies on some $b^2 x - a^2 y = \pm ab$ with $\gcd(a, b) = 1$.

This completes the proof.

14.3. Remarks (Useful Parameter Choices on a Column)

If 4 divides x, the square height $y = \frac{x}{4}$ yields $x * \left(\frac{x}{4}\right) = \left(\frac{x}{2}\right)^2$.

If $x = 2s$ with odd $s \ge 5$, $y = \frac{s-1}{2}$ yields the oblong $xy = s(s - 1)$.

If x has a square factor $a^2 \ge 9$, taking $y = \frac{x}{a^2}$ covers the column by squares; for odd squares $x = a^2 \ge 9$, the height $y = 4$ always works.

14.4. Consequences For the Classifier

Combining the corollary (odd primes not covered) with the theorem above (all composites except $x = 4$ are covered) yields a perfect classifier within the range $2 \le y \le x - 2$: a column x is covered if and only if x is composite, with the single exception $x = 4$. Odd prime columns ($p \ge 5$) remain uncovered within this range."

15. Geometric Presence and Absence of Squares in the TMT

The analysis of square presence in the TMT reveals where geometric symmetry first arises in the Full Multiplication Table (FMT): along the diagonal line $x = y$, which corresponds to the locus $x * y = n^2$. This line also marks the transition where the two complementary divisors of each product exchange order between the largest and smallest factors.

Each square intersection or product $x * y = n^2$ sets up the limiting case where a divisor pair (d_1, d_2) becomes symmetric along the diagonal $x = y$. Together with the oblong family $x * y = n(n \pm 1)$, these two bundles decide all quadratic products that take part in the sieve mechanism within the admissible range $2 \leq y \leq x - 2$. Since every composite column is covered by at least one square or oblong intersection, the presence of uncovered columns above the base line $(y = 1)$, hence of primes, must arise precisely from those squares and oblongs whose intersections fall outside this permitted range.

In this sense, the prime phenomenon appears geometrically because of absent quadratic configurations within the TMT lattice. The present section focuses on the square family, while the next one extends the same reasoning to the oblong domain, completing the square-oblong duality that underlies the coverage criterion.

15.1. Motivation and Setting

In the Triangular Multiplication Table (TMT), the geometric locus defined by $xy = n^2$ forms one of the two principal quadratic bundles that govern the table's structure. The point where the diagonal $x = y$ meets the square hyperbola $x * y = n^2$ is the square intersection corresponding to n^2. As with oblongs, square presence is decided by the existence or absence of such intersections within the admissible triangular domain of the TMT, $2 \leq y \leq x - 2$. When no integer y in this range satisfies $xy = n^2$, the corresponding column x is said to exclude any square intersections within its admissible vertical range.

15.2. Algebraic Condition for Square Visibility

Fix a product $x * y = n^2$ with $x \geq y$. Solving for y gives $y = \frac{n^2}{x}$. To belong to the admissible TMT range, this must satisfy $2 \leq \frac{n^2}{x} \leq x - 2$, or equivalently, $x^2 - 2x - n^2 \geq 0$. Solving for x yields $x \geq 1 + \sqrt{n^2 + 1}$. For a fixed value of n, this inequality specifies the smallest column index at which the square hyperbola $x * y = n^2$ first intersects the admissible triangular region $2 \leq y \leq x - 2$. For smaller x, the intersection lies above the boundary $y = x - 2$ and therefore outside the admissible domain; only once $x \geq 1 + \sqrt{n^2 + 1}$ does the corresponding square point become geometrically visible within the TMT.

15.3. Theorem (Presence of Square Products in the TMT)

Fix $n \geq 1$. There exist integers x, y satisfying $2 \leq y \leq x - 2$ and $x * y = n^2$ if and only if there exists a divisor x of n^2 such that $1 + \sqrt{n^2 + 1} \leq x \leq \frac{n^2}{2}$, and then necessarily $y = \frac{n^2}{x}$.

Proof:

Suppose $x * y = n^2$ with $2 \le y \le x - 2$. Without loss of generality, take $x \ge y$, so x is the largest complementary divisor of the product $d_1 * d_2 = n^2$. From $y \le x - 2$ and $y = \frac{n^2}{x}$, we have $\frac{n^2}{x} \le x - 2 \Leftrightarrow x^2 - 2x - n^2 \ge 0 \Leftrightarrow x \ge 1 + \sqrt{n^2 + 1}$. From $y \ge 2$ we obtain $\frac{n^2}{x} \ge 2 \Leftrightarrow x \le \frac{n^2}{2}$. Since x divides n^2 and $y = \frac{n^2}{x}$, both inequalities are satisfied, setting up the claimed bounds.

Conversely, let x divide n^2 and satisfy $1 + \sqrt{n^2 + 1} \le x \le \frac{n^2}{2}$, and set $y = \frac{n^2}{x}$. The upper bound ensures $y \ge 2$. The lower bound implies $x^2 - 2x - n^2 \ge 0$, which is equivalent to $y \le x - 2$. Thus $2 \le y \le x - 2$ and $x * y = n^2$, as needed.

15.4. Remarks

1. The interval is nonempty only if $1 + \sqrt{n^2 + 1} \le \frac{n^2}{2}$, which fails for $n = 1$ and $n = 2$. Hence no square appears in the admissible range for $n = 1$ or $n = 2$.
2. For $n = 3$, the interval is $\left[1 + \frac{\sqrt{10,9}}{2}\right) \approx [4.16, 4.5)$, but no divisor x of 9 lies in that range, so there is still no admissible intersection.
3. For $n = 4$, the interval is $\left[1 + \sqrt{17}, 8\right) \approx [5.12, 8]$, and $x = 8$ divides 16, giving the point $(x, y) = (8,2)$ with $2 \le 2 \le 6$, confirming the first admissible square intersection.

15.5. Classification of Excluded Squares

The presence condition also shows the cases of absence, which occur when n^2 has no divisor x satisfying $1 + \sqrt{n^2 + 1} \le x \le \frac{n^2}{2}$. This happens precisely when n is a prime number, not a composite. For such n, the product n^2 cannot be realized as $x * y$ within the admissible range $2 \le y \le x - 2$. Its only lattice representation lies on the diagonal point (n, n). Hence, prime squares p^2 are the minimal examples of excluded square products, marking the first threshold points on the TMT diagonal. By contrast, the columns $x = p^2$ are fully covered by square intersections but have no oblongs.

15.6. Corollary (Prime-Square Columns Contain Only Square Intersections)

For every prime $p \ge 3$, the column $x = p^2$ admits square intersections and no oblong intersections within the admissible range $2 \le y \le x - 2$. No oblong intersections occur. Indeed, for any integer $y = t^2$, the product $x * y = p^2 t^2 = (p * t)^2$ is a perfect square, and since $2 \le t^2 \le p^2 - 2$ for $t \ge 2$ and $p \ge 3$, these intersections all lie strictly below the top point (p, p).

15.7. Composite Square Visibility

In contrast, when n is composite, n^2 admits interior square representations in the TMT. For instance, with $n = 6$ we have $n^2 = 36 = 9 * 4$, which lies strictly below the diagonal $x = y$ and satisfies $2 \leq y \leq x-2$. In general, write $n = ab$ with $a \geq 2$ and $b \geq 2$. Then the factor pair $(x, y) = (ab^2, a)$ satisfies $xy = (ab^2)a = a^2b^2 = (ab)^2 = n^2$, with $2 \leq y = a$ and $x - y = a(b^2-1) \geq 3a \geq 6$, hence $2 \leq y \leq x-2$. Since $\frac{x}{y} = b^2 \geq 4$, the point lies strictly below the diagonal. This construction works for all composite n, including prime powers.

15.8. Proposition (Composite Squares Are Fully Visible)

If n is composite, the product n^2 yields at least one admissible lattice point (x, y) in the TMT with $2 \leq y \leq x-2$ such that $x * y = n^2$. For every composite $n \geq 4$, there exists an interior square intersection strictly below the diagonal $x = y$.

Proof:

Write $n = ab$ with integers $a \geq 2$ and $b \geq 2$. Set $y := a$ and $x := ab^2$. Then $xy = (ab^2)a = a^2b^2 = (a * b)^2 = n^2$. Moreover, $y = a \geq 2$ and $x - y = a(b^2-1) \geq 3a \geq 6$, so $2 \leq y \leq x-2$. Finally, $x \geq y$ and $x > y$ because $b \geq 2$, hence the point lies strictly below $x = y$. This completes the proof.

Note that even the smallest composite square, $4^2 = 16$, admits an interior intersection at $(x, y) = (8,2)$, confirming that the statement holds for all composite $n \geq 4$.

15.9. Geometric Summary and Interpretation

The family of square hyperbolas $xy = n^2$ generates a bundle of parabolic curves sharing the axis of symmetry $x = 4y$, as established in Section 11. Their lower intersections populate the interior of the TMT whenever n is composite, while prime squares stay confined to the boundary $x = y$. The absence of prime squares thus forms the geometric counterpart of the arithmetic constraint on prime columns: primes and their squares generate uncovered verticals in the interior band.

Geometrically, the excluded squares lie exactly on the top edges of their columns, forming the uppermost layer of the TMT, composite squares appear below, marking the first visible intersections of the square bundle within the admissible domain.

15.10. Square Families Presence (Composite Squares) and Absence (Prime Squares) in the Coverage of the X-Axis within the TMT

Within the TMT lattice, which behaves as a sieve of primes restricted to the admissible range $2 \leq y \leq x-2$, the products xy that yield perfect squares, either visible in the range (presence) or absent (absence), fall into two complementary families within the global sequence of perfect squares (A000290).

The squares of composite numbers (A062312) correspond to visible products $x * y = n^2$ within the admissible range. They cover precisely the composite columns $x \in \{$ A246547 (prime-power composites) $\cup$ A126706 (mixed composites) $\}$, each column admitting at least one square product n^2 below the diagonal $x = y$.

In contrast, the squares of primes (A001248) correspond to excluded products $x * y = n^2$ that never intersect any column $x \in \{$ A002808 $\}$ within the admissible height range. These prime-square products appear only along the diagonal $x = y$ at the topmost point $(x, y) = (p, p)$ and have no secondary square intersections below.

The distinction between composite and prime squares sets up the subset of $x * y = n^2$ products that take part in the TMT coverage criterion. Composite squares generate interior intersections that occur along composite x-columns, while prime squares appear only at the diagonal point (p, p), never covering any x-column within the admissible range $2 \leq y \leq x - 2$.

See Figure C003589 in section 11.1 for the parabolic loci of squares in the TMT, showing the distinct geometric placement of composite square products below $x = y$ line.

16. Geometric Presence and Absence of Oblongs in the TMT

The analysis of oblong presence completes the geometric framework started by the square loci $x * y = n^2$. While the squares define the symmetric boundary where a divisor pair (d_1, d_2) coincides along $x = y$, the oblongs describe the adjacent configurations that occur one step above or below that symmetry. Their parabolic envelopes share the same axis x=4y but introduce a distinct arithmetic modulation governed by the identity $1 + 4 * oblong = (odd\ square)$. In the TMT lattice, the oblong loci occupy precisely those positions left vacant by the square curves, jointly ensuring that every composite column is reached by at least one quadratic intersection. Consequently, the uncovered columns, those devoid of both square and oblong intersections, correspond exactly to the prime columns. Thus, the oblong family provides the complementary half of the square-oblong sieve, completing the geometric mechanism that explains primality through absence rather than construction.

16.1. Motivation and Setting

In the Triangular Multiplication Table (TMT), each lattice point (x, y) is the product $x * y$ within the bounded domain $2 \leq y \leq x - 2$. Among these products, special attention is given to the oblong numbers $g = n(n + 1)$, which form the sequence of integers found between consecutive perfect squares. An oblong number is said to be present in the TMT if there exist integers x and y satisfying $2 \leq y \leq x - 2$ and $x * y = g$ with x odd; otherwise, it is said to be absent. Finding which oblongs appear in this lattice is equivalent to asking for which n the product $n(n + 1)$ can be represented within the admissible triangular range of the TMT.

16.2. Algebraic Condition for Oblong Visibility

Fix a product $x * y = n(n \pm 1)$ with $x \geq y$. Solving for y gives $y = \frac{n(n\pm1)}{x}$. To belong to the admissible TMT range, this must satisfy $2 \leq \frac{n(n\pm1)}{x} \leq x - 2$, or equivalently, $x^2 - 2x - n(n \pm 1) \geq 0$. Solving for x yields $x \geq 1 + \sqrt{n(n \pm 1) + 1}$. For a fixed value of n, this inequality specifies the smallest column index at which the oblong hyperbola $xy = n(n \pm 1)$ first intersects the admissible triangular range $2 \leq y \leq x - 2$. For smaller x, the intersection lies above the boundary $y = x - 2$ and therefore outside the admissible domain; only once $x \geq 1 +$

$\sqrt{n(n \pm 1) + 1}$ does the corresponding oblong point become geometrically visible within the TMT.

16.3. Theorem (Presence Condition for Oblongs in the TMT)

Let $g = n(n+1)$ and let d_2 be its largest odd divisor. Then g is realizable within the odd-column range $2 \le y \le x - 2$ if and only if $d_2 \ge 1 + \sqrt{g+1}$; otherwise, it is absent.

Proof:

Starting from the defining relation $x * y = g = n(n+1)$ with $x \ge y$, the product belongs to the admissible TMT range only if $2 \le y \le x - 2$. Substituting $y = \frac{g}{x}$ gives the compound inequality $2 \le \frac{g}{x} \le x - 2$. Multiplying through by x yields $2x \le g \le x^2 - 2x$. The right-hand inequality can be rewritten as $x^2 - 2x - g \ge 0$, which implies that x must satisfy $x \ge 1 + \sqrt{g+1}$.

This condition expresses the minimal horizontal coordinate at which the oblong hyperbola $x * y = g$ can intersect the admissible triangular range of the TMT. For smaller x the intersection lies above the boundary $y = x - 2$, outside the range $2 \le y \le x - 2$.

To ensure that both x and y are integers, x must be a divisor of g. Since the TMT odd columns correspond to odd x, only the odd divisors of g are relevant. Among them, the largest odd divisor d_2 gives the maximal possible x within this parity constraint. Hence an oblong intersection exists within the admissible range if and only if this maximal odd divisor satisfies $d_2 \ge 1 + \sqrt{g+1}$.

If this inequality holds, one can choose $x = d_2$ and $y = \frac{g}{d_2}$, obtaining $2 \le y \le x - 2$ and thus a valid lattice point (x, y) representing g within the TMT. Conversely, if $d_2 < 1 + \sqrt{g+1}$, then all admissible $x \le d_2$ fall below the threshold needed to satisfy $x^2 - 2x - g \ge 0$, meaning that g lies entirely above the triangular range and therefore no lattice intersection exists within the TMT.

Therefore, $g = n(n+1)$ is realizable in the TMT if and only if $d_2 \ge 1 + \sqrt{g+1}$; otherwise, it is absent.

16.4. Corollary (Characterization via Powers of Two)

The only oblong numbers that do not appear within the admissible triangular range of the TMT, $2 \le y \le x - 2$, are those lying at once next to powers of two. Explicitly, an oblong $g = n(n+1)$ is absent if and only if $n = 2^k - 1$ or $n = 2^k$ for some integer $k \ge 1$. Equivalently, the absent oblongs are precisely those of the form $g = 4^k - 2^k$ and $g = 4^k + 2^k$ for $k \ge 1$.

Proof:

Let $g = n(n+1)$ and d_2 be its largest odd divisor. From the presence condition $d_2 \ge 1 + \sqrt{g+1}$, an oblong is realizable in the TMT if and only if this inequality holds.

If $n = 2^k - 1$, then $g = 2^k(2^k - 1)$ and $d_2 = 2^k - 1$. Substituting into the inequality gives $2^k - 1 < 1 + \sqrt{2^k(2^k - 1) + 1}$, which fails for all $k \ge 1$.

Similarly, if $n = 2^k$, then $g = 2^k(2^k + 1)$ and $d_2 = 2^k + 1$. Again $2^k + 1 < 1 + \sqrt{2^k(2^k + 1) + 1}$ for every $k \geq 1$.

In both cases, the largest odd divisor is too small to satisfy the visibility threshold, preventing any lattice intersection within the range $2 \leq y \leq x - 2$.

Conversely, for all other values of n, the largest odd divisor $d_2 \geq 1 + \sqrt{g + 1}$, ensuring at least one admissible intersection. Hence, the oblongs of the form $4^k \pm 2^k$ are the only absent ones.

16.5. Classification of Absent Oblongs

The characterization above directly shows two infinite families of absent oblongs. These correspond to cases in which n or $n + 1$ is a power of two, so that the product $g = n(n + 1)$ has no sufficiently large odd divisor to satisfy $d_2 \geq 1 + \sqrt{g + 1}$.

Case A: If $n = 2^k - 1$, then $g = 2^k(2^k - 1)$ and the largest odd divisor is $d_2 = 2^k - 1$. Substituting into the inequality gives $d_2 < 1 + \sqrt{g + 1} = 1 + \sqrt{2^k(2^k - 1) + 1}$, which fails for all $k \geq 1$. Hence all such numbers are absent.

Case B: If $n = 2^k$, then $g = 2^k(2^k + 1)$ with largest odd divisor $d_2 = 2^k + 1$. Again $d_2 < 1 + \sqrt{g + 1} = 1 + \sqrt{2^k(2^k + 1) + 1}$, and the inequality fails for all $k \geq 1$. Hence these numbers are also absent.

Both cases yield oblongs whose odd part is too small relative to their total size to allow intersection with the TMT. The absent oblongs therefore form two complementary sequences symmetric with respect to the even powers of two:

A390325 Oblong numbers of the form 2^k*(2^k - 1) or 2^k*(2^k + 1). 0

2, 6, 12, 20, 56, 72, 240, 272, 992, 1056, 4032, 4160, 16256, 16512, 65280, 65792, 261632, 262656, 1047552, 1049600, 4192256, 4196352, 16773120, 16781312, 67100672, 67117056, 268419072, 268451840, 1073709056, 1073774592, 4294901760, 4295032832, 17179738112, 17180000256

(list; graph; refs; listen; history; published version; edit; text; internal format)

OFFSET	1,1
COMMENTS	Oblong numbers whose distance from a perfect square of the form 4^k (A000302) is 2^k (A000079), forming the interleaved pairs 4^k - 2^k (A020522) and 4^k + 2^k (A063376). These are the oblong numbers for which no pair (x, y) with x odd satisfies x*y = m*(m + 1) and 2 <= y <= floor(x/4).
LINKS	Table of n, a(n) for n=1..34. Index entries for linear recurrences with constant coefficients, signature (0,6,0,-8).
FORMULA	a(2n - 1) = 2^n*(2^n - 1) = 4^n - 2^n = A020522(n). a(2n) = 2^n*(2^n + 1) = 4^n + 2^n = A063376(n). a(n) = 2^(ceiling(n/2))*(2^(ceiling(n/2)) + (-1)^n) = 4^(ceiling(n/2)) + (-1)^n*2^(ceiling(n/2)). Complement of A390326 within A002378.
EXAMPLE	For k = 3, (2^3-1)*2^3 = 7*8 = 56, and 2^3*(2^3+1) = 8*9 = 72. Then, 56 and 72 form the consecutive oblong pair surrounding the square (2^3)^2 = 64, and the square 64 is equidistant from 56 and 72 at a distance of 2^3. For k = 4, (2^4-1)*2^4 = 15*16 = 240, and 2^4*(2^4+1) = 16*17 = 272. Then, 240 and 272 form the consecutive oblong pair surrounding the square (2^4)^2 = 256, and the square 256 is equidistant from 240 and 272 at a distance of 2^4.
CROSSREFS	Cf. A390326 (complementary oblong numbers), A002378 (oblong numbers), A000079 (powers of 2), A000302 (powers of 4), A020522 (oblongs 4^k - 2^k), A063376 (oblongs 4^k + 2^k). Sequence in context: A290209 A286780 A099885 * A106372 A354895 A214916 Adjacent sequences: A390318 A390319 A390322 * A390327 A390328 A390329
KEYWORD	nonn,easy,changed
AUTHOR	Charles Kusniec, Nov 01 2025
STATUS	proposed

16.6. Composite and Oblong Presence

For all other values of n, the inequality $d_2 \geq 1 + \sqrt{g + 1}$ holds, and the corresponding oblong numbers are fully represented within the TMT. These realizable oblongs correspond to columns that intersect the oblong hyperbolas $x * y = n(n + 1)$ within the range $2 \leq y \leq x - 2$.

Proposition (All Other Oblongs Are Present):

If $n \neq 2^k - 1$ and $n \neq 2^k$, then $g = n(n+1)$ can be expressed as $x * y$ within the triangular domain $2 \leq y \leq x - 2$ with x odd.

Examples.

The oblongs 30, 42, 90, 110, and 132 all satisfy the inequality with wide margin since their largest odd divisors d_2 are much greater than $\sqrt{g+1}$, confirming full presence within the TMT.

A390326 Oblong numbers not of the forms 2^k*(2^k - 1) or 2^k*(2^k + 1). 0

30, 42, 90, 110, 132, 156, 182, 210, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 1122, 1190, 1260, 1332, 1406, 1482, 1560, 1640, 1722, 1806, 1892, 1980, 2070, 2162, 2256, 2352, 2450, 2550, 2652, 2756, 2862, 2970, 3080, 3192, 3306, 3422, 3540, 3660, 3782, 3906, 4290, 4422, 4556, 4692, 4830, 4970, 5112, 5256

(list; graph; refs; listen; history; published version; edit; text; internal format)

OFFSET	1,1
COMMENTS	Oblong numbers whose distance from any perfect square of the form 4^k (A000302) is not 2^k (A000079), consisting of all oblongs (A002378) except the consecutive pairs 4^k - 2^k (A020522) and 4^k + 2^k (A063376). These are the oblong numbers for which there exists a pair (x, y) with x odd satisfying x*y = m*(m + 1) and 2 <= y <= floor(x/4).
LINKS	Table of n, a(n) for n=1..60.
FORMULA	{ m*(m + 1) : m >= 1 } \ { 2^k*(2^k - 1), 2^k*(2^k + 1) : k >= 1 }. A002378 \ (A020522 union A063376). Complement of A390325 within A002378.
EXAMPLE	Between 4^2 = 16 and 4^3 = 64, the oblong numbers are 20 = 4*5, 30 = 5*6, 42 = 6*7, 56 = 7*8. Only 20 = 4^2 + 2^2 and 56 = 4^3 - 2^3 are at a distance of 2^t from a perfect square of the form 4^k. Then, 20 and 56 do not belong in this sequence. The others {30, 42} have distances, from any perfect square 4^k, not of the form 2^t and therefore belong to this sequence. Between 4^3 = 64 and 4^4 = 256, the oblong numbers are 72 = 8*9, 90 = 9*10, 110 = 10*11, 132 = 11*12, 156 = 12*13, 182 = 13*14, 210 = 14*15, 240 = 15*16. Only 72 = 4^3 + 2^3 and 240 = 4^4 - 2^4 are at a distance of 2^t from a perfect square of the form 4^k. Then, 72 and 240 do not belong in this sequence. The others {90, 110, 132, 156, 182, 210} have distances, from any perfect square 4^k, not of the form 2^t and therefore belong to this sequence.
CROSSREFS	Cf. A390325 (complementary oblong numbers), A002378 (oblong numbers), A000079 (powers of 2), A000302 (powers of 4), A020522 (oblongs 4^k - 2^k), A063376 (oblongs 4^k + 2^k). Sequence in context: A382873 A376800 A257832 * A050776 A268697 A258358 Adjacent sequences: A390318 A390319 A390322 * A390327 A390328 A390329
KEYWORD	nonn,changed
AUTHOR	Charles Kusniec, Nov 01 2025
STATUS	proposed

16.7. Geometric Summary and Interpretation

The family of oblong hyperbolas $x * y = g = n(n \pm 1)$ forms a bundle of parabolic curves sharing the same axis $x = 4y$ as the square family but shifted by the constant term that encodes the relation $1 + 4g = (odd\ square)$. These curves fill the gaps left by the square hyperbolas, creating a perfectly interlaced pattern within the TMT lattice. The absent oblongs correspond to the uppermost parabolas that do not intersect any lattice point in $2 \leq y \leq x - 2$, while the realizable oblongs populate the interior range, complementing the visible squares described in the earlier section.

Geometrically, this structure implies that the combined families $xy = n^2$ and $xy = n(n \pm 1)$ together account for all quadratic intersections across the TMT. Columns covered by at least one of them correspond to composite integers, while columns devoid of both stay uncovered—precisely the prime columns. Thus, the oblongs provide the geometric counterpart to the squares, completing the dual coverage mechanism that defines the sieve nature of the TMT.

16.8. OEIS Classification

The results above yield a complete partition of the oblong sequence A002378 into two complementary subsets:

Absent oblongs in TMT: $g \in \{2^k(2^k - 1), 2^k(2^k + 1)\} =$ A390325.

Present (realizable) oblongs in TMT: $g \in$ A002378 \ A390325 = A390326.

Hence A002378 = A390325 ∪ A390326, with A390325 ∩ A390326=∅.

Geometrically, the absent oblongs lie just outside the admissible boundary of the TMT, while the realizable oblongs appear as interior intersections within $2 \leq y \leq x - 2$, perfectly complementing the visible square lattice.

See Figure xxxxxx for the visual placement of absent and realizable oblongs in the TMT.

Acknowledgments

I extend my deepest gratitude to H. Bli Shem and my family for their unwavering support and constant inspiration throughout this journey. As a Talmud proverb profoundly states, "*He who does not teach his son a trade teaches him to steal.* Among the wisest of trades is the pursuit of knowledge through study. I dedicate these studies to my beloved children.(מ.ז.ל.)

References

[1] Kusniec, C. (2024, March 02). The Quadratic Distribution of Integers Relative to Squares and Oblongs. https://www.researchgate.net/publication/379989460_The_Quadratic_Distribution_of_Integers_Relative_to_Squares_and_Oblongs

[2] Michael De Vlieger, "Omega Multiplicity Diagram(for the ω/Ω taxonomy). https://www.vincico.com/radix/omegamultiplicity.html

[3] OEIS A246547 (composite prime powers), A120944 (composite squarefree), A126706 (composites with a squared prime factor and at least one other distinct prime), A002808 (composites).

[4] Kusniec, C. (2025). The Missing Divisor and the Four-Zone Prime Theorem: Resolving Classical Prime Conjectures. Zenodo. https://doi.org/10.5281/zenodo.16889967

[5] Kusniec, C. (2025). Fundamental Theorem of Composite Numbers. Zenodo. https://doi.org/10.5281/zenodo.17382001